Mathematical Methods

IIT-JAM-2005

OBJECTIVE QUESTIONS

Q1. Which of the following is ***INCORRECT*** for the matrix $M = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

(a) It is its own inverse
(b) It is its own transpose
(c) It is non-orthogonal
(d) It has eigen values ± 1

Ans. : (c)

Solution: The inverse of the given matrix is $M^{-1} = \frac{1}{|M|}\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} = \frac{1}{(-1)}\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = M$

Thus the given matrix is its own inverse.

The transpose of M is, $M^T = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = M$

The given matrix is orthogonal as each row vector is a unit vector and the two rows are orthogonal.

The eigenvalues of orthogonal matrix are $+1$ or -1. For the given matrix

$$\lambda_1 + \lambda_2 = 0 \Rightarrow \lambda_1 = 1 \text{ and } \lambda_2 = -1$$

Thus option (c) is correct option.

Q2. A periodic function can be expressed in a Fourier series of the form,

$f(x) = \sum_{n=0}^{\infty}\left(a_n \cos(nx) + b_n \sin(nx)\right)$. The functions $f_1(x) = \cos^2 x$ and $f_2(x) = \sin^2 x$ are expanded in their respective Fourier series. If the coefficients for the first series are $a_n^{(1)}$ and $b_n^{(1)}$, and the coefficients for the second series are $a_n^{(2)}$ and $b_n^{(2)}$, respectively, then which of the following is correct?

(a) $a_2^{(1)} = \frac{1}{2}$ *and* $b_2^{(2)} = \frac{-1}{2}$
(b) $b_2^{(1)} = \frac{1}{2}$ *and* $a_2^{(2)} = \frac{-1}{2}$
(c) $a_2^{(1)} = \frac{1}{2}$ *and* $a_2^{(2)} = \frac{-1}{2}$
(d) $b_2^{(1)} = \frac{1}{2}$ *and* $b_2^{(2)} = \frac{-1}{2}$

Ans. : (c)

Solution: $f_1(x)=\frac{1}{2}+\frac{1}{2}\cos 2x$ and $f_2(x)=\frac{1}{2}-\frac{1}{2}\cos 2x$

Hence, $a_2^{(1)}=\frac{1}{2}$ and $a_2^{(2)}=-\frac{1}{2}$

All the b_n's of each of the series are zero. As there is no sine terms in any of the two given functions.

Thus the correct option is (c).

IIT-JAM 2006

Q3. The symmertric part of $P=\begin{pmatrix} a \\ b \end{pmatrix}(a-2b)$ is

(a) $\begin{pmatrix} a^2-2 & ba-1 \\ ba-1 & b^2-2 \end{pmatrix}$

(b) $\begin{pmatrix} a(a-2) & b \\ b & b^2 \end{pmatrix}$

(c) $\begin{pmatrix} a(a-1) & b(a-1) \\ b(a-1) & b^2 \end{pmatrix}$

(d) $\begin{pmatrix} a(a-2) & b(a-1) \\ b(a-1) & b^2 \end{pmatrix}$

Ans. : (d)

Solution: The given matrix can be written as $P=\begin{pmatrix} a \\ b \end{pmatrix}(a-2 \quad b)=\begin{pmatrix} a(a-2) & ab \\ b(a-2) & b^2 \end{pmatrix}$

The transpose of P is,

$$P^T=\begin{pmatrix} a(a-2) & b(a-2) \\ ab & b^2 \end{pmatrix}$$

Hence the symmetric part of P is,

$$\frac{P+P^T}{2}=\frac{1}{2}\begin{pmatrix} 2a(a-2) & 2ab-2b \\ 2ab-2b & 2b^2 \end{pmatrix}=\begin{pmatrix} a(a-2) & b(a-1) \\ b(a-1) & b^2 \end{pmatrix}$$

Hence the correct option is (d).

IIT-JAM 2007

Q4. $\begin{pmatrix} x & y \end{pmatrix}\begin{pmatrix} 5 & -7 \\ 7 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 15$

The matrix equation of above represents

(a) a circle of radius $\sqrt{15}$ (b) an ellipse of semi major axis $\sqrt{5}$

(c) an ellipse of semi major axis 5 (d) a hyperbola

Ans. : (b)

Solution: $\begin{pmatrix} x & y \end{pmatrix}\begin{pmatrix} 5 & -7 \\ 7 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 15 \Rightarrow \begin{pmatrix} x & y \end{pmatrix}\begin{pmatrix} 5x-7y \\ 7x+3y \end{pmatrix} = 15$

$$\Rightarrow 5x^2 - 7xy + 7xy + 3y^2 = 15 \Rightarrow 5x^2 + 3y^2 = 15 \Rightarrow \frac{x^2}{3} + \frac{y^2}{5} = 1$$

Thus the given equation represents an ellipse with semi-major axis $\sqrt{5}$.

Q5. $f(x)$ is a periodic function of x with a period of 2π. In the interval $-\pi < x < \pi$, $f(x)$ is given by

$$f(x) = \begin{cases} 0, & -\pi < x < 0 \\ \sin x, & 0 < x < \pi \end{cases}$$

In the expansion of $f(x)$ as a Fourier series of sine and cosine functions, the coefficient of $\cos(2x)$ is

(a) $\frac{2}{3\pi}$ (b) $\frac{1}{\pi}$ (c) 0 (d) $-\frac{2}{3\pi}$

Ans. : (d)

Solution: The coefficients of $\cos 2x$ is a_2.

Thus, $a_2 = \frac{1}{\pi}\left[\int_{-\pi}^{0} 0 \cdot \cos 2x dx + \int_{0}^{\pi} \sin x \cdot \cos 2x dx\right]$

$$\Rightarrow a_2 = \frac{1}{2\pi}\int_0^{\pi} (\sin 3x - \sin x) dx = \frac{1}{2\pi}\left[-\frac{\cos 3x}{3} + \cos x\right]_0^{\pi}$$

$$= \frac{1}{2\pi}\left[\left(-\frac{\cos 3x}{3} + \cos \pi\right) - \left(-\frac{1}{3} + 1\right)\right] = \frac{1}{2\pi}\left[\frac{1}{3} - 1 + \frac{1}{3} - 1\right] = -\frac{2}{3\pi}$$

IIT-JAM 2008

Q6. The product PQ of any two real, symmetric matrices P and Q is

(a) symmetric for all P and Q (b) never symmetric

(c) symmetric, if $PQ = QP$ (d) anti-symmetric for all P and Q

Ans. : (c)

Solution: A matrix is symmetric, if its transpose is equal to the matrix itself.

Hence for the matrix PQ, $(PQ)^T = Q^T P^T$ (since $(AB)^T = B^T A^T$)

Since, Q and P are symmetric matrices; $Q^T = Q, P^T = P$

Hence, $(PQ)^T = QP$

It is easily seen that $(PQ)^T$ will be equal to PQ, only if $QP = PQ$. Hence (c) is correct option.

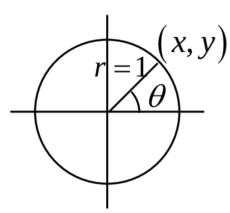
Q7. The work done by a force in moving a particle of mass m from any point (x, y) to a neighboring point $(x+dx,\ y+dy)$ is given by $dW = 2xydx + x^2dy$. The work done for a complete cycle around a unit circle is

(a) 0 (b) 1 (c) 3 (d) 2π

Ans. : (a)

Solution: Let us write the co-ordinates x and y as,

$x = (1)\cos\theta, y = (1)\sin\theta \Rightarrow x = \cos\theta$ and $y = \sin\theta$.



unit circle

Thus, $dx = -\sin\theta\, d\theta$ and $dy = \cos\theta\, d\theta$

Thus, the given work dW can be written as,

$$dW = 2(\cos\theta)(\sin\theta)(-\sin\theta)d\theta + (\cos\theta)^2\cos\theta\, d\theta$$

$$= -2\sin^2\theta \cdot \cos\theta\, d\theta + \cos^3\theta\, d\theta$$

Thus the total work done along the complete circle is

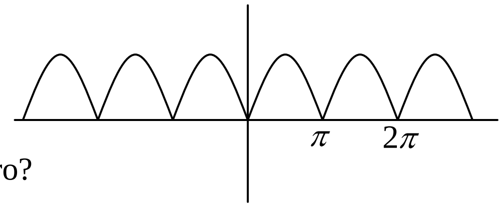
$$W = -2\int_0^{2\pi} \sin^2\theta \cdot \cos\theta\, d\theta + \int_0^{2\pi} \cos^3\theta\, d\theta$$

It can be easily checked that the value of each of these integrals is 0. Hence, the correct option is (a).

IIT-JAM 2009

Q8. In the Fourier series of the periodic function (shown in the figure)

$$f(x)=|\sin x|$$
$$=\sum_{n=0}^{\infty}\left[\alpha_n \cos nx+\beta_n \sin nx\right]$$



Which of the following coefficients are non-zero?

(a) α_n for odd n (b) α_n for even n

(c) β_n for odd n (d) β_n for even n

Ans. : (b)

Solution: The given function is an even function (assuming the basic interval of definition to be symmetric about the origin)

Hence, all the B_n' s are 0.

$$\alpha_n=\frac{2}{\pi}\int_0^{\pi}|\sin x|\cos nx\,dx=\frac{2}{\pi}\int_0^{\pi}\sin x\cdot\cos nx\,dx$$

$$=\frac{2}{2\pi}\int_0^{\pi}\left[\sin(n+1)x-\sin(n-1)x\right]dx=\frac{1}{\pi}\left[\frac{-\cos(n+1)x}{(n+1)}+\frac{\cos(n-1)x}{(n-1)}\right]_0^{\pi}$$

For odd n,

$$\alpha_n=\frac{1}{\pi}\left[-\frac{1}{(n+1)}+\frac{1}{(n-1)}+\frac{1}{(n+1)}-\frac{1}{(n-1)}\right]\Rightarrow\alpha_n=0\text{ , for odd }n.$$

For even n,

$$\alpha_n=\frac{1}{\pi}\left[\frac{1}{(n+1)}-\frac{1}{(n-1)}+\frac{1}{(n+1)}-\frac{1}{(n-1)}\right]\Rightarrow\alpha_n=\frac{2}{\pi}\left[\frac{1}{(n+1)}-\frac{1}{(n-1)}\right]$$

$$=\frac{2}{\pi}\left[\frac{n-1-n-1}{(n^2-1)}\right]=-\frac{4}{\pi(n^2-1)}$$

Thus for even n, α_n is nonzero. Hence the correct option is (b).

IIT-JAM 2010

Q9. A matrix is given by $M=\frac{1}{\sqrt{2}}\begin{pmatrix} i & 1 \\ 1 & i \end{pmatrix}$. The eigenvalues of M are

(a) real and positive

(b) purely imaginary with modulus 1

(c) complex with modulus 1

(d) real and negative

Ans. : (c)

Solution: We know that if λ is an eigenvalue of matrix A, then $k\lambda$ is the eigenvalue of matrix kA. Hence Let as evaluate the eigenvalue of matrix

$$M'=\begin{pmatrix} i & 1 \\ 1 & i \end{pmatrix}$$

For the calculation of eigenvalues

$$\begin{vmatrix} i-\lambda & 1 \\ 1 & i-\lambda \end{vmatrix}=0\Rightarrow (i-\lambda)^2-1=0\Rightarrow (i-\lambda)=\pm 1$$

$$\Rightarrow \lambda=i+1,\ i-1$$

Thus the eigenvalues of the given matrix M are

$$\lambda_1=\frac{1}{\sqrt{2}}(1+i)=\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i \text{ and } \lambda_2=\frac{1}{\sqrt{2}}(i-1)=\frac{-1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i$$

We see that $|\lambda_1|=|\lambda_2|=1$. Thus the correct option is (c).

Q10. The equation of a surface of revolution is $z=\pm\sqrt{\frac{3}{2}x^2+\frac{3}{2}y^2}$. The unit vector normal to the surface at the point $A\left(\sqrt{\frac{2}{3}},0,1\right)$ is

(a) $\sqrt{\frac{3}{5}}\hat{i}+\frac{2}{\sqrt{10}}\hat{k}$ (b) $\sqrt{\frac{3}{5}}\hat{i}-\frac{2}{\sqrt{10}}\hat{k}$ (c) $\sqrt{\frac{3}{5}}\hat{i}+\frac{2}{\sqrt{5}}\hat{k}$ (d) $\sqrt{\frac{3}{10}}\hat{i}+\frac{2}{\sqrt{10}}\hat{k}$

Ans. : (b)

Solution: $z=\pm\sqrt{\frac{3}{2}x^2+\frac{3}{2}y^2}\Rightarrow z^2=\frac{3}{2}x^2+\frac{3}{2}y^2\Rightarrow 3x^2+3y^2-2z^2=0$

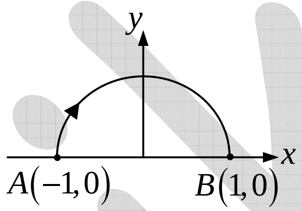
Let $V=3x^2+3y^2-2z^2$, taking gradient $\Rightarrow \vec{\nabla}V=6x\hat{x}+6y\hat{y}-4z\hat{z}$.

The unit normal to the surface at the point $A\left(\sqrt{\frac{2}{3}},0,1\right)$ is $\hat{n}=\frac{\vec{\nabla}V}{\left|\vec{\nabla}V\right|}$. Thus

$$\hat{n}=\frac{6\sqrt{\frac{2}{3}}\hat{x}+6\times 0\hat{y}-4\times 1\hat{z}}{\sqrt{36\times\frac{2}{3}+16}}=\frac{6\sqrt{\frac{2}{3}}\hat{x}-4\hat{z}}{\sqrt{40}}=\sqrt{\frac{3}{5}}\hat{x}-\frac{2}{\sqrt{10}}\hat{z}$$

IIT-JAM 2011

Q11. The line integral $\int_A^B \vec{F}\cdot d\vec{l}$, where $\vec{F}=\frac{x}{\sqrt{x^2+y^2}}\hat{x}+\frac{y}{\sqrt{x^2+y^2}}\hat{y}$, along the semi-circular path as shown in the figure below is



(a) -2 (b) 0 (c) 2 (d) 4

Ans. : (b)

Solution: $x^2+y^2=1\Rightarrow xdx=-ydy$ and $d\vec{l}=dx\hat{x}+dy\hat{y}$

$$\Rightarrow \vec{F}.d\vec{l}=\frac{xdx}{\sqrt{x^2+y^2}}+\frac{ydy}{\sqrt{x^2+y^2}}=0\Rightarrow\int_A^B\vec{F}\cdot d\vec{l}=0 \qquad (\because xdx=-ydy)$$

Q12. Given two $(n\times n)$ matrices $\hat{P}$ and $\hat{Q}$ such that $\hat{P}$ is Hermitian and $\hat{Q}$ is skew (anti)-Hermitian. Which one of the following combinations of $\hat{P}$ and $\hat{Q}$ is necessarily a Hermitian matrix?

(a) $\hat{P}\hat{Q}$ (b) $i\hat{P}\hat{Q}$ (c) $\hat{P}+i\hat{Q}$ (d) $\hat{P}+\hat{Q}$

Ans.: (c)

Solution: Any matrix is hermitian if its conjugate transpose is equal to the matrix itself.

For, $\hat{P}\hat{Q}$, we have $\left(\hat{P}\hat{Q}\right)^*=\left(\hat{Q}\right)^*\left(\hat{P}\right)^*=\left(-\hat{Q}\right)\left(\hat{P}\right)=-\hat{Q}\hat{P}$

Thus, $\hat{P}\hat{Q}$ is not hermitian.

For matrix $i\hat{P}\hat{Q}$, we have $\left(i\hat{P}\hat{Q}\right)=(-i)\left(\hat{P}\hat{Q}\right)^*=(-i)\left(\hat{Q}\right)\left(\hat{P}\right)=-i\hat{Q}\hat{P}$

Thus, $i\hat{P}\hat{Q}$ is not hermitian.

For matrix $\hat{P}+i\hat{Q}$, we have

$$\left(\hat{P}+i\hat{Q}\right)^* = \left(\hat{P}\right)^* + \left(i\hat{Q}\right)^* = \hat{P}+(-i)\left(\hat{Q}\right)^* = \hat{P}+(-i)\left(-\hat{Q}\right) = \hat{P}+i\hat{Q}$$

Thus $\hat{P}+i\hat{Q}$ is hermitian.

For $\left(\hat{P}+\hat{Q}\right)$, we have $\left(\hat{P}+\hat{Q}\right)^* = \left(\hat{P}\right)^* + \left(\hat{Q}\right)^* = \hat{P}-\hat{Q}$

Thus, $\left(\hat{P}+\hat{Q}\right)$ is not hermitian.

Note: In this question "*" symbol has been used to denote the conjugate transpose of a matrix.

IIT-JAM-2012

Q13. If $\vec{F}$ is a constant vector and $\vec{r}$ is the position vector then $\vec{\nabla}\left(\vec{F}\cdot\vec{r}\right)$ would be

(a) $\left(\vec{\nabla}\cdot\vec{r}\right)\vec{F}$ (b) $\vec{F}$ (c) $\left(\vec{\nabla}\cdot\vec{F}\right)\vec{r}$ (d) $|\vec{r}|\vec{F}$

Ans.: (b)

Solution: Let $\vec{F} = F_0\left(\hat{x}+\hat{y}+\hat{z}\right)$ and $\vec{r} = x\hat{x}+y\hat{y}+z\hat{z} \Rightarrow \vec{F}.\vec{r} = F_0\left(x+y+z\right)$.

Thus $\vec{\nabla}\left(\vec{F}\cdot\vec{r}\right) = F_0\left(\hat{x}+\hat{y}+\hat{z}\right) = \vec{F}$

IIT-JAM-2013

Q14. The inverse of the matrix

$$M = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \text{ is}$$

(a) $M-I$ (b) M^2-I (c) $I-M^2$ (d) $I-M$

where I is the identity matrix.

Ans.: (b)

Solution: Given $M = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$

The characteristics equation is,

$$\begin{vmatrix} -\lambda & 1 & 1 \\ 0 & -\lambda & 1 \\ 1 & 0 & -\lambda \end{vmatrix} = 0$$

$$-\lambda\left(\lambda^2 - 0\right) - (-1) + 1(\lambda) = 0 \Rightarrow -\lambda^3 + \lambda + 1 = 0 \Rightarrow \lambda^3 - \lambda - 1 = 0$$

Thus the cayley-Hamilton theorem gives

$$M^3 - M - I = 0$$

Multiply both sides by M^{-1} gives

$M^2 - I - M^{-1} = 0 \Rightarrow M^{-1} = M^2 - I$. Thus option (b) is correct option.

Q15. The value of $\sqrt{i} + \sqrt{-i}$, where $i = \sqrt{-1}$, is

(a) 0 (b) $\frac{1}{\sqrt{2}}$ (c) $\sqrt{2}$ (d) $-\sqrt{2}$

Ans.: (a)

Solution: $$\sqrt{i} + \sqrt{-i} = \frac{\left(\sqrt{i} + \sqrt{-i}\right)\left(\sqrt{i} - \sqrt{-i}\right)}{\sqrt{i} - \sqrt{-i}} = \frac{i - \sqrt{-i^2} + \sqrt{-i^2} - \sqrt{i^2}}{\sqrt{i} - \sqrt{-i}}$$

$$= \frac{i - \sqrt{-i}}{\sqrt{i} - \sqrt{-i}} = \frac{i - i}{\sqrt{i} - \sqrt{-i}} = 0$$

Q16. The solution of the differential equation $dz(x,y) + xz(x,y)dx + yz(x,y)dy = 0$ is.........

Ans.: $Ce^{-\left(x^2+y^2\right)/2}$

Given differential equation can be written as,

$$dz(x,y) + z(x,y)\left[xdx + ydy\right] = 0 \Rightarrow \frac{dz}{z} = -xdx - ydy$$

Integrating both sides gives

$$\ln z = -\frac{x^2}{2} - \frac{y^2}{2} + \ln c \Rightarrow \ln\frac{z}{c} = \frac{-\left(x^2 + y^2\right)}{2}$$

$$\Rightarrow \frac{z}{c} = e^{-\left(x^2+y^2\right)/2} \Rightarrow z = ce^{-\left(x^2+y^2\right)/2}.$$

Q17. Given that $f(1)=1, f'(1)=1,$ and $f''(1)=1$, the value of $f(1/2)$ is …………..

Ans.: 0.606

Solution: Let $f(x)=ke^x$

In order to satisfy each of the three given conditions $k=\frac{1}{e}$.

Thus $f(x)=\frac{e^x}{e}$

Hence, $f(1/2)=\frac{e^{1/2}}{e}=\frac{1}{\sqrt{e}}=0.606$.

IIT-JAM-2014

Q18. For vectors $\vec{a}=\hat{j}+\hat{k}, \vec{b}=2\hat{i}+3\hat{j}-5\hat{k}$ and $\vec{c}=\hat{j}-\hat{k}$, the vector product $\vec{a}\times(\vec{b}\times\vec{c})$ is

(a) in the same direction as $\vec{c}$

(b) in the direction opposite to $\vec{c}$

(c) in the same direction as $\vec{b}$

(d) in the direction opposite to $\vec{b}$

Ans.: (a) $\vec{b}\times\vec{c}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 2 & 3 & -5\\ 0 & 1 & -1\end{vmatrix}=\hat{i}(-3+5)-\hat{j}(-2+0)+\hat{k}(2-0)=2\hat{i}+2\hat{j}+2\hat{k}$

$$\vec{a}\times(\vec{b}\times\vec{c})=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 0 & 1 & 1\\ 2 & 2 & 2\end{vmatrix}=\hat{i}(2-2)-\hat{j}(0-2)+\hat{k}(0-2)=2\hat{j}-2\hat{k}=2\vec{c}$$

Two Marks

Q19. The value of $\sum_{n=0}^{\infty} r^n \sin(n\theta)$ for r = 0.5 and $\theta=\frac{\pi}{3}$ is

(a) $\frac{1}{\sqrt{3}}$

(b) $\sqrt{\frac{2}{3}}$

(c) $\sqrt{\frac{3}{2}}$

(d) $\sqrt{3}$

Ans.: (a)

Solution: $\sum_{n=0}^{\infty} r^n \sin(n\theta)=0+r\sin\theta+r^2\sin 2\theta+r^3+\sin 3\theta+\cdots$

Let, $Z=re^{i\theta}=r(\cos\theta+i\sin\theta)$

$\Rightarrow Z^2 = r^2 e^{i2\theta} = r^2\left(\cos 2\theta + i\sin 2\theta\right)$ and so on.

Thus, we can see,

$$\sum_{n=0}^{\infty} r^n \sin\left(n\theta\right) = \text{Img part of } \left\{\sum_{n=0}^{\infty} z^n\right\}$$

$$\sum_{n=0}^{\infty} z^n = \frac{z}{1-z} = \frac{re^{i\theta}}{1-re^{i\theta}} = \frac{e^{i\pi/3}/2}{1-\frac{e^{i\pi/3/2}}{2}} = \frac{1/2\left[\cos 60^0 + i\sin 60^0\right]}{1-\frac{1}{2}\left[\cos 60^0 + i\sin 60^0\right]}$$

$$= \frac{1/2\left[\frac{1}{2}+i\frac{\sqrt{3}}{2}\right]}{1-\frac{1}{2}\left[\frac{1}{2}+i\frac{\sqrt{3}}{2}\right]} = \frac{\frac{1}{4}+i\frac{\sqrt{3}}{4}}{\frac{3}{4}-\frac{i\sqrt{3}}{4}} = \frac{1+i\sqrt{3}}{3-i\sqrt{3}} = \frac{\left(1+i\sqrt{3}\right)\left(3+i\sqrt{3}\right)}{\left(3-i\sqrt{3}\right)\left(3+i\sqrt{3}\right)}$$

$$\sum_{n=0}^{\infty} z^n = \frac{\left(3-3\right)}{12} + \frac{i\sqrt{3}+3i\sqrt{3}}{12} = \frac{i4\sqrt{3}}{12}$$

Thus, $\sum_{n=0}^{\infty} r^n \sin\left(n\theta\right) = \frac{1}{\sqrt{3}}$

Q20. If the surface integral of the field $\vec{A}(x,y,z) = 2\alpha x\hat{i} + \beta y\hat{j} - 3\gamma z\hat{k}$ over the closed surface of an arbitrary unit sphere is to be zero, then the relationship between α, β and γ is

(a) $\alpha + \beta/6 - \gamma = 0$ (b) $\alpha/3 + \beta/6 - \gamma/2 = 0$

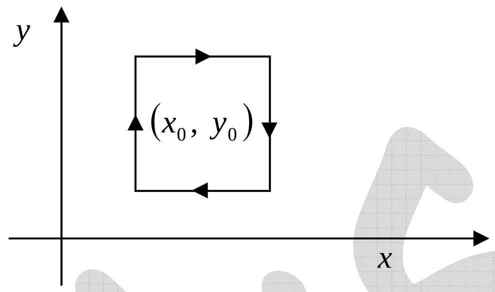
(c) $\alpha/2 + \beta - \gamma/3 = 0$ (d) $2/\alpha + 1/\beta - 3/\gamma = 0$

Ans.: (b)

Solution: It is given that $\oint_S \vec{A}.d\vec{a} = 0 \Rightarrow \int_V \left(\vec{\nabla}\cdot\vec{A}\right)d\tau = 0$ (From Divergence Theorem)

$$\int_V \left(\vec{\nabla}.\vec{A}\right)d\tau = 0 \Rightarrow 2\alpha + \beta - 3\gamma = 0 \Rightarrow \frac{\alpha}{3} + \frac{\beta}{6} - \frac{\gamma}{2} = 0$$

Q21. The line integral $\oint \vec{A}\cdot d\vec{l}$ of a vector field $\vec{A}(x,\ y)=\frac{1}{r^2}\left(-y\hat{i}+x\hat{j}\right)$ where $r^2=x^2+y^2$ is taken around a square (see figure) of side of unit length and centered at $\left(x_0,y_0\right)$ with $\left|x_0\right|>\frac{1}{2}$ and $\left|y_0\right|>\frac{1}{2}$. If the value of the integral is L, then



(a) L depends on $\left(x_0,y_0\right)$

(b) L is independent of $\left(x_0,y_0\right)$ and its value is -1

(c) L is independent of $\left(x_0,y_0\right)$ and its value is 0

(d) L is independent of $\left(x_0,y_0\right)$ and its value is 2

Ans.: (c)

Solution: $\vec{\nabla}\times\vec{A}=\begin{pmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -\frac{y}{x^2+y^2} & \frac{x}{x^2+y^2} & 0 \end{pmatrix}$

$$=\hat{x}(0-0)-\hat{y}(0-0)+\hat{z}\left[\frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right)+\frac{\partial}{\partial y}\left(\frac{y}{x^2+y^2}\right)\right]$$

$$\vec{\nabla}\times\vec{A}=\hat{z}\left[\frac{\left(x^2+y^2\right)-x\times 2x}{\left(x^2+y^2\right)^2}+\frac{\left(x^2+y^2\right)-y\times 2y}{\left(x^2+y^2\right)^2}\right]=\hat{z}\left[\frac{\left(y^2-x^2\right)+x^2-y^2}{\left(x^2+y^2\right)^2}\right]=0$$

Thus, $\oint \vec{A}\cdot d\vec{l}=0$.

IIT-JAM-2015

Q22. Consider the coordinate transformation $x' = \frac{x+y}{\sqrt{2}}$, $y' = \frac{x-y}{\sqrt{2}}$. The relation between the area elements $dx'dy'$ and $dxdy$ is given by $dx'dy' = j\,dxdy$. The value of j is

(a) 2 (b) 1 (c) -1 (d) -2

Ans.: (c)

Solution: $x' = \frac{x+y}{\sqrt{2}}$, $y' = \frac{x-y}{\sqrt{2}}$

$$\because \quad dx'dy' = J\,dxdy \Rightarrow J = \begin{vmatrix} \frac{\partial x'}{\partial x} & \frac{\partial x'}{\partial y} \\ \frac{\partial y'}{\partial x} & \frac{\partial y'}{\partial y} \end{vmatrix} = \begin{vmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{vmatrix} = -\frac{1}{2} - \frac{1}{2} = -1$$

Q23. The trace of a 2×2 matrix is 4 and its determinant is 8. If one of the eigenvalues is $2(1+i)$, the other eigenvalue is

(a) $2(1-i)$ (b) $2(1+i)$ (c) $(1+2i)$ (d) $(1-2i)$

Ans.: (a)

Solution: $\lambda_1 = 2+2i,\ \lambda_2 = 2(1-i) \Rightarrow \lambda_1 + \lambda_2 = 4$ and $\lambda_1 \cdot \lambda_2 = 8$

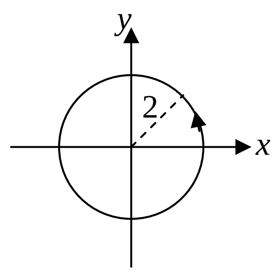
Two Marks:

Q24. Consider a vector field $\vec{F} = y\hat{i} + xz^3\hat{j} - zy\hat{k}$. Let C be the circle $x^2 + y^2 = 4$ on the plane $z = 2$, oriented counter-clockwise. The value of the contour integral $\oint_C \vec{F}\cdot d\vec{r}$ is

(a) 28π (b) 4π (c) -4π (d) -28π

Ans.: (a)

Solution: $\because \quad \oint_C \vec{F}.d\vec{r} = \int_S \left(\vec{\nabla}\times\vec{F}\right).d\vec{a}$



$$\vec{\nabla}\times\vec{F} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y & xz^3 & -zy \end{vmatrix}$$

$$\Rightarrow \vec{\nabla}\times\vec{F} = \hat{x}\left(\frac{\partial(-yz)}{\partial y} - \frac{\partial\left(xz^3\right)}{\partial z}\right) + \hat{y}\left(\frac{\partial y}{\partial z} - \frac{\partial(-zy)}{\partial x}\right) + \hat{z}\left(\frac{\partial\left(xz^3\right)}{\partial x} - \frac{\partial y}{\partial y}\right)$$

$$\Rightarrow \vec{\nabla}\times\vec{F} = \hat{x}\left(-z-3xz^2\right)+\hat{y}\left(0-0\right)+\hat{z}\left(z^3-1\right)$$

$$\because z=2 \Rightarrow \vec{\nabla}\times\vec{F} = -\left(2+12x\right)\hat{x}+7\hat{z}$$

$$\because d\vec{a} = rdrd\phi\hat{z} \Rightarrow \left(\vec{\nabla}\times\vec{F}\right).d\vec{a} = \left[-\left(2+12x\right)\hat{x}+7\hat{z}\right].rdrd\phi\hat{z} = 7rdrd\phi$$

$$\Rightarrow \int_S\left(\vec{\nabla}\times\vec{F}\right).d\vec{a} = 7\int_0^2 rdr\int_0^{2\pi}d\phi = 28\pi$$

Q25. Consider the equation $\frac{dy}{dx}=\frac{y^2}{x}$ with the boundary condition $y(1)=1$. Out of the following the range of x in which y is real and finite is

(a) $-\infty \le x \le -3$ (b) $-3 \le x \le 0$ (c) $0 \le x \le 3$ (d) $3 \le x \le \infty$

Ans.: (d)

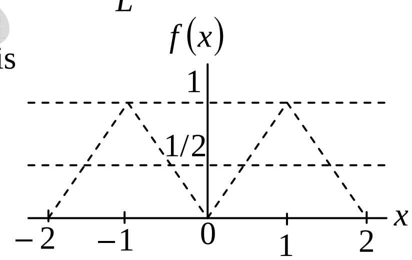
Solution: $\frac{dy}{dx}=\frac{y^2}{x} \Rightarrow \frac{dy}{y^2}=\frac{dx}{x} \Rightarrow -\frac{1}{y}=\ln x + C'$

$$y(1)=1 \Rightarrow -\frac{1}{1}=\ln 1 + C' \Rightarrow C'=-1 \Rightarrow -\frac{1}{y}=\ln x - 1 \Rightarrow y=\frac{1}{1-\ln x}$$

At $x=0, y=\infty$ and $\ln x$ is not defined for negative values of x.

Thus, correct option is (d).

Q26. The Fourier series for an arbitrary periodic function with period $2L$, is given by $f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos\frac{n\pi x}{L}+\sum_{n=1}^{\infty}b_n\sin\frac{n\pi x}{L}$. For the particular periodic function shown in the figure the value of a_0 is



(a) 0 (b) 0.5 (c) 1 (d) 2

Ans.: (c)

Solution: The wavefunction of the given function can be written as

$$f(x)=\begin{cases} x & 0<x<1 \\ -x & -1<x<0 \end{cases}$$

Coefficient a_0 is defined as

$$a_0 = 1\int_{-1}^{0} -x\,dx + 1\int_{0}^{1} x\,dx = -\left[\frac{x^2}{2}\right]_{-1}^{0} + \left[\frac{x^2}{2}\right]_{0}^{1} = -\left[0 - \frac{(-1)^2}{2}\right] + \left[\frac{(1)^2}{2} - 0\right] = +\frac{1}{2} + \frac{1}{2} - 1$$

$\therefore a_0 = 1$

Q27. The phase of the complex number $(1+i)i$ in the polar representation is

(a) $\frac{\pi}{4}$ (b) $\frac{\pi}{2}$ (c) $\frac{3\pi}{4}$ (d) $\frac{5\pi}{4}$

Ans.: (c)

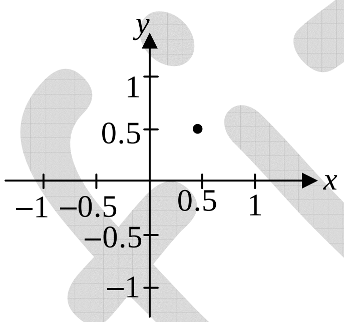
Solution: $z = (1+i)i \Rightarrow z = (-1+i)$ for $z = x + iy$

$$\tan\theta = \frac{y}{x} = -1 \Rightarrow \theta = \tan^{-1}(-1) \Rightarrow \theta = \frac{3\pi}{4}$$

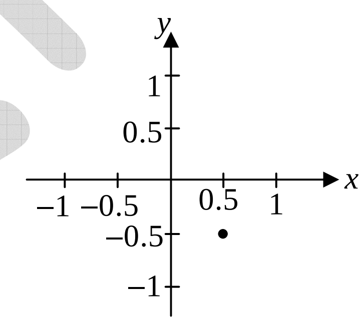
IIT-JAM-2016

Q28. Which of the following points represent the complex number $= \frac{1}{1-i}$?

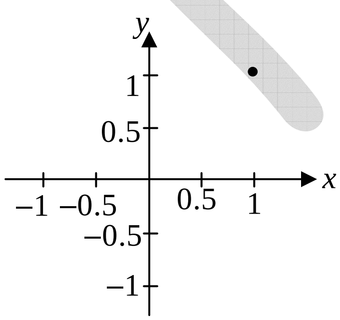
(a)



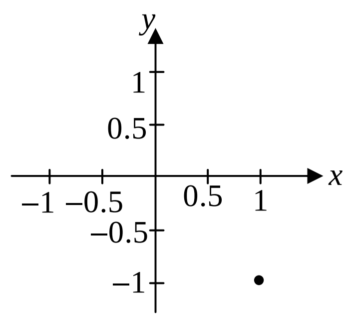
(b)



(c)



(d)



Ans.: (a)

Solution: $\frac{1}{1-i} = \frac{1}{1-i} \times \left(\frac{1+i}{1+i}\right) = \frac{1+i}{1+1} = \frac{1}{2} + \frac{1}{2}i$

Q29. The eigenvalues of the matrix representing the following pair of linear equations

$$x+iy=0$$
$$ix+y=0$$

are

(a) $1+i,\ 1+i$ (b) $1-i,\ 1-i$ (c) $1,\ i$ (d) $1+i,\ 1-i$

Ans.: (d)

Solution: Characteristic equation is $|A-\lambda I|=0$

$$\Rightarrow \begin{vmatrix} 1-\lambda & i \\ i & 1-\lambda \end{vmatrix}=0 \Rightarrow (1-\lambda)^2-i^2 \Rightarrow (1-\lambda)^2+1=0 \Rightarrow (1-\lambda)=\pm i \Rightarrow \lambda=1+i,\ \ 1-i$$

Q30. For the given set of equations

$$x+y=1,\ \ y+z=1,\ \ x+z=1,$$

which one of the following statements is correct?

(a) Equations are inconsistent

(b) Equations are consistent and a single non-trivial solution exists

(c) Equations are consistent and many solutions exist

(d) Equations are consistent and only a trivial solution exists.

Ans.: (b)

Solution: The augmented matrix of the system can be written as $M=\begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix}$

Row reduction gives $M \approx \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & -1 & 1 & 0 \end{bmatrix} \approx \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 2 & 1 \end{bmatrix}$

Thus, $x+y=1,\ \ y+z=1$ and $2z=1$

The last equation gives $z=1/2$. Using first two equations we find $x=y=1/2$. Thus the system has a single non trivial solution. The correct option is (b)

Q31. The tangent line to the curve $x^2+xy+5=0$ at $(1,1)$ is represented by

(a) $y=3x-2$ (b) $y=-3x+4$

(c) $x=3y-2$ (d) $x=-3y+4$

Ans.: (b)

Solution: Given $x^2+xy+5=0 \Rightarrow 2x+y+x\frac{dy}{dx}=0 \Rightarrow \frac{dy}{dx}=\frac{-(2x+y)}{x}$

At $(1,1)$, $\frac{dy}{dx}=-\frac{3}{1}=-3$

Hence the equation of tangent line is $y-1=-3(x-1) \Rightarrow y=-3x+4$

Q32. Fourier series of a given function $f(x)$ in the interval 0 to L is

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi nx}{L}\right)+\sum_{n=1}^{\infty} b_n \sin\left(\frac{2\pi nx}{L}\right).$$

If $f(x)=x$ in the region $(0,\pi), b_2=$ ………………

Ans.: $-0\cdot 5$

Solution: Here, $2l=\pi \Rightarrow l=\pi/2$

$$b_2=\frac{2}{\pi}\int_0^{\pi} x\sin\frac{2\pi x}{\pi/2}dx=\frac{2}{\pi}\int_0^{\pi} x\sin 4x dx=\frac{2}{\pi}\left[\frac{-x\cos 4x}{4}+\frac{1}{16}\sin 4x\right]_0^{\pi}$$

$$=\frac{2}{\pi}\left[\left(\frac{-\pi\cos\pi}{4}+\frac{1}{16}\sin 4\pi-0\right)\right]=\frac{2}{\pi}\left(\frac{-\pi}{4}\right)=-\frac{1}{2}=-0\cdot 5.$$

Q33. Consider a function $f(x,y)=x^3+y^3$, where y represents a parabolic curve x^2+1. The total derivative of f with respect to x, at $x=1$ is…………………

Ans.: 27

Solution: $f(x,y)=x^3+y^3$. Also given that, $y=x^2+1$

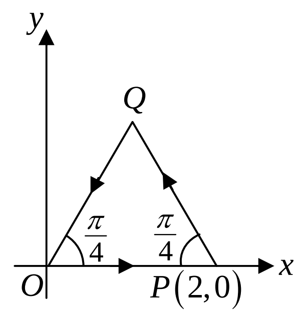
Hence, $f(x,y)=f(x)=x^3+(x^2+1)^3$

$$\therefore \quad \frac{df(x,y)}{dx}=\frac{df(x)}{dx}=3x^2+3(x^2+1)^2 2x$$

Hence, the total derivative at $x=1$ is $3(1)+3(1^2+1)^2\cdot 2\cdot 1=3+6\cdot 4=27$

Q34. Consider a closed triangular contour traversed in counter-clockwise direction, as shown in the figure.

The value of the integral, $\oint \vec{F}\cdot\vec{dl}$ evaluated along this contour, for a vector field, $\vec{F} = y\hat{e}_x - x\hat{e}_y$, is………….. ($\hat{e}_x, \hat{e}_y$ and $\hat{e}_z$ are unit vectors in Cartesian-coordinate system).



Ans.: -2

Solution: $\because \vec{F} = y\hat{e}_x - x\hat{e}_y \Rightarrow \vec{\nabla}\times\vec{F} = -2\hat{z}$ and $d\vec{a} = dxdy\hat{z} \Rightarrow \left(\vec{\nabla}\times\vec{F}\right).d\vec{a} = -2dxdy$

$$\oint \vec{F}\cdot\vec{dl} = \int\left(\vec{\nabla}\times\vec{F}\right).d\vec{a} = \int\int(-2)dxdy = (-2)\frac{1}{2}(2\times1) = -2$$

Q35. A hemispherical shell is placed on the x-y plane centered at the origin. For a vector field $\vec{E} = \left(-y\hat{e}_x + x\hat{e}_y\right)/\left(x^2+y^2\right)$, the value of the integral $\int_S\left(\vec{\nabla}\times\vec{E}\right)\cdot d\vec{a}$ over the hemispherical surface is……………….π.

($d\vec{a}$ is the elemental surface area, $\hat{e}_x, \hat{e}_y$ and $\hat{e}_z$ are unit vectors in Cartesian-coordinate system)

Ans.: 2

Solution: $\vec{E} = \left(-y\hat{e}_x + x\hat{e}_y\right)/\left(x^2+y^2\right) \Rightarrow \vec{\nabla}\times\vec{E} = 0$ except at origin.

$$\because \quad \int_S\left(\vec{\nabla}\times\vec{E}\right).d\vec{a} = \oint_{line}\vec{E}.d\vec{l}$$

We have to take line integral around circle $x^2+y^2=r^2$ in $z=0$ plane. Let us use cylindrical coordinate and use $x = r\cos\phi,\ y = r\sin\phi \Rightarrow dx = -r\sin\phi d\phi,\ dy = r\cos\phi d\phi$.

$$\vec{E}.d\vec{l} = \left(-ydx + xdy\right)/\left(x^2+y^2\right) = \frac{-r\sin\phi\left(-r\sin\phi\right)d\phi + r\cos\phi\left(r\cos\phi\right)d\phi}{r^2} = d\phi$$

$$\Rightarrow \oint_{line}\vec{E}.d\vec{l} = \int_0^{2\pi}d\phi = 2\pi$$

IIT-JAM 2017

Q36. For the three matrices given below, which one of the choices is correct?

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_2 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

(a) $\sigma_1\sigma_2 = -i\sigma_3$ (b) $\sigma_1\sigma_2 = i\sigma_3$ (c) $\sigma_1\sigma_2 + \sigma_2\sigma_1 = I$ (d) $\sigma_3\sigma_2 = -i\sigma_1$

Ans. : (b)

Solution: These are pauli spin matrix which will satisfied $\sigma_1\sigma_2 = i\sigma_3$ and $\sigma_1\sigma_2 + \sigma_2\sigma_1 = 0$

Q37. The integral of the vector $\vec{A}(\rho,\varphi,z) = \frac{40}{\rho}\cos\varphi\,\hat{\rho}$ (standard notation for cylindrical coordinates is used) over the volume of a cylinder of height L and radius R_0 is:

(a) $20\pi R_0 L(\hat{i}+\hat{j})$ (b) 0 (c) $40\pi R_0 L\,\hat{j}$ (d) $40\pi R_0 L\,\hat{i}$

Ans. : (d)

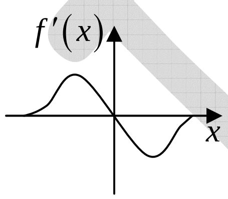
Solution: By seeing the options lets calculate

$$\int_V \vec{A}d\tau = \int_0^{R_0}\int_0^{2\pi}\int_0^{L} \frac{40}{\rho}\cos\varphi\left(\cos\varphi\hat{i} + \sin\varphi\hat{j}\right)\rho\, d\rho d\varphi dz \qquad \because \ \hat{\rho} = \cos\phi\hat{i} + \sin\phi\hat{j}$$

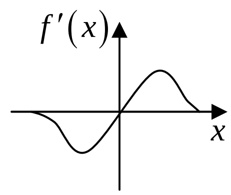
$$\Rightarrow \int_V \vec{A}d\tau = 40R_0L\int_0^{2\pi}\cos\varphi\left(\cos\varphi\hat{i} + \sin\varphi\hat{j}\right)d\varphi = 40R_0L\pi\hat{i}$$

Q38. Which one of the following graphs represents the derivative $f'(x) = \frac{df}{dx}$ of the function $f(x) = \frac{1}{1+x^2}$ most closely (graphs are schematic and not drawn to scale)?

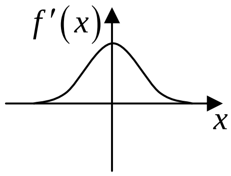
(a)



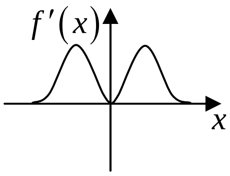
(b)



(c)

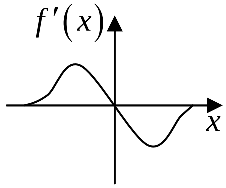


(d)



Ans. : (a)

Solution: $f(x)=\frac{1}{1+x^2}$ and $f'(x)=\frac{df}{dx}=\frac{-2x}{1+x^2}$ anti-symmetric function but $f(-x)$ is positive and $f(x)$ is positive



Q39. For the Fourier series of the following function of period 2π

$$f(x)=\begin{cases}0 & -\pi<x<0\\ 1 & 0<x<\pi\end{cases}$$

the ratio (to the nearest integer) of the Fourier coefficients of the first and the third harmonic is:

(a) 1 (b) 2 (c) 3 (d) 6

Ans. : (c)

Solution: $a_0=\frac{1}{2\pi}\int_0^{\pi}(1)\,dx=\frac{1}{2\pi}\cdot\pi=\frac{1}{2}$

$$a_n=\frac{1}{\pi}\int_0^{\pi}(1)\cos nx\,dx=\frac{1}{n\pi}\left[\sin nx\right]_0^{\pi}=0$$

$$b_n=\frac{1}{\pi}\int_0^{\pi}(1)\sin nx\,dx=-\frac{1}{n\pi}\left[\cos nx\right]_0^{\pi}=\left(-\frac{1}{n\pi}\right)(-1-1)=\frac{2}{n\pi}$$

Hence, $\frac{b_1}{b_3}=\frac{2}{\pi}\times\frac{3\pi}{2}=3$

Q40. The volume integral of the function $f(r,\theta,\phi)=r^2\cos\theta$ over the region $(0\le r\le 2,\ 0\le\theta\le\pi/3 \text{ and } 0\le\phi\le 2\pi)$ is............

(Specify your answer upto two digits after the decimal point)

Ans. : 15.07

Solution: $I=\int_V f\,d\tau=\int_0^{2}\int_0^{\pi/3}\int_0^{2\pi}\left(r^2\cos\theta\right)r^2\sin\theta\,dr\,d\theta\,d\phi=\frac{2^5}{5}\frac{1}{2}\left[-\frac{\cos 2\theta}{2}\right]_0^{\pi/3}\times 2\pi$

$$\Rightarrow I=\frac{32}{5}\frac{1}{4}\left[-\cos 2\pi/3+\cos 0\right]_0^{\pi/3}\times 2\pi=\frac{24}{5}\pi=15.07$$

Q41. Consider two particles moving along the x - axis. In terms of their coordinates x_1 and x_2, their velocities are given as $\frac{dx_1}{dt} = x_2 - x_1$ and $\frac{dx_2}{dt} = x_1 - x_2$, respectively. When they start moving from their initial locations of $x_1(0) = 1$ and $x_2(0) = -1$, the time dependence of both x_1 and x_2 contains a term of the form e^{at}, where a is a constant. The value of a (an integer) is………………

Ans. : 2

Solution: From the given relations we can write

$$\frac{dx_1}{dt} = -\frac{dx_2}{dt}$$

Integrating both sides with respect to t gives, $x_1 = -x_2 + c_1$, where c being a constant of integration

At $t = 0, x_1 = 1$ and $x_2 = -1$

Hence, $c = 0$

Thus, $x_1 = -x_2$ (i)

Using equation (i) the first equation can be written as

$$\frac{dx_1}{dt} = -2x_1 \Rightarrow \frac{dx_1}{x_1} = -2dt$$

$$\Rightarrow \ln x_1 = -2t + \ln k_1 \Rightarrow \ln \frac{x_1}{k_1} = -2t \Rightarrow x_1(t) = k_1 e^{-2t}$$

Using $x_1(0) = 1$, we obtain $k_1 = 1$, thus $x_1 = e^{-2t}$

Using equation (ii) the second equation can be written as

$$\frac{dx_2}{dt} = -2x_2 \Rightarrow \frac{dx_2}{x_2} = -2dt$$

Integrating gives

$$\ln x_2 = -2t + \ln k_2 \Rightarrow \ln \frac{x_2}{k_2} = e^{-2t}$$

Thus, $x_2 = k_2 e^{-2t}$

Using $x_2(0)=-1$, we obtain $k_2=-1$

Thus, $x_2(t)=-e^{-2t}$

Hence, the value of a is -2.

Q42. Consider the differential equation $y''+2y'+y=0$. If $y(0)=0$ and $y'(0)=1$, then the value of $y(2)$ is..................

(Specify your answer to two digits after the decimal point)

Ans. : 0.27

Solution: The characteristic equation is $m^2+2m+1=0 \Rightarrow (m+1)^2=0$

Thus $m=-1$ is a repeated root

Thus the general solution is

$$y=(c_1+c_2x)e^{-x}$$

since $y(0)=0 \Rightarrow 0=c_1 \Rightarrow c_1=0$

Thus we can write $y=c_2xe^{-x} \Rightarrow y'=c_2\left(e^{-x}-xe^{-x}\right)$

since $y'(0)=1$

$1=c_2(1-0) \Rightarrow c_2=1$

$y=xe^{-x}$

$$y(2)=2e^{-2}=\frac{2}{e^2}=\frac{2}{(2.72)^2}=0.27$$

IIT-JAM 2018

Q43. Let $f(x,y)=x^3-2y^3$. The curve along which $\nabla^2 f=0$ is

(a) $x=\sqrt{2}y$ (b) $x=2y$

(c) $x=\sqrt{6}y$ (d) $x=\frac{-y}{2}$

Ans.: (b)

Solution: $\nabla^2 f=\frac{\partial^2}{\partial x^2}\left(x^3-2y^3\right)+\frac{\partial^2}{\partial y^2}\left(x^3-2y^3\right)+0$

$$\nabla^2 f=6x-12y$$

$$\because \nabla^2 f=0 \Rightarrow 6x-12y=0 \Rightarrow x=2y$$

Q44. A curve is given by $\vec{r}(t)=t\hat{i}+t^2\hat{j}+t^3\hat{k}$. The unit vector of the tangent to the curve at $t=1$ is

(a) $\frac{\hat{i}+\hat{j}+\hat{k}}{\sqrt{3}}$ (b) $\frac{\hat{i}+\hat{j}+\hat{k}}{\sqrt{6}}$ (c) $\frac{\hat{i}+2\hat{j}+2\hat{k}}{3}$ (d) $\frac{\hat{i}+2\hat{j}+3\hat{k}}{\sqrt{14}}$

Ans.: (d)

Solution: Let $\hat{n}$ be a unit vector tangent to the curve at t.

$$\hat{n}=\frac{d\vec{r}/dt}{|d\vec{r}/dt|}=\frac{\hat{i}+2t\hat{j}+3t\hat{k}}{\sqrt{1+4t^2+5t^2}} \Rightarrow \text{ at } t=1, \hat{n}=\frac{\hat{i}+2j+3\hat{k}}{\sqrt{14}}$$

Q45. The function $f(x)=\begin{cases} x, & -\pi<x<0 \\ -x, & 0<x<\pi \end{cases}$ is expanded as a Fourier series of the form $a_0+\sum_{n=1}^{\infty} a_n \cos(nx)+\sum_{n=1}^{\infty} b_n \sin(nx)$. Which of the following is true?

(a) $a_0\neq 0, b_n=0$ (b) $a_0\neq 0, b_n\neq 0$

(c) $a_0=0, b_n=0$ (d) $a_0=0, b_n\neq 0$

Ans.: (b)

Solution:- $f(x)=\begin{cases} x, & -\pi<x<0 \\ -x, & 0<x<\pi \end{cases}$

$$a_0=\frac{1}{2\pi}\left\{\int_{-\pi}^{0} xdx+\int_0^{\pi} -xdx\right\}=\frac{-\pi}{2} \Rightarrow a_0\neq 0$$

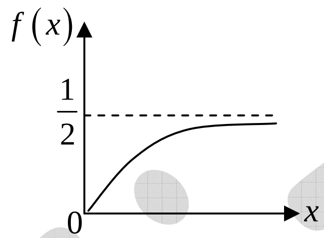
$$b_n = \frac{1}{\pi}\left\{\int_{-\pi}^{0} x\sin nx dx + \int_{0}^{\pi} -x\sin nx dx\right\}$$

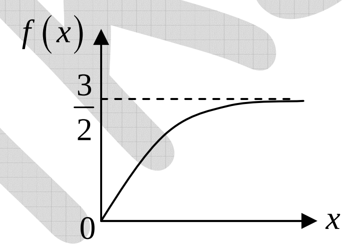
$$= \frac{1}{\pi}\left\{\left(\frac{-x\cos nx}{n} + \frac{\sin nx}{n^2}\right)\Bigg|_{-\pi}^{0} \left(\frac{x\cos nx}{n} + \frac{\sin nx}{n^2}\right)\Bigg|_{0}^{\pi}\right\}$$

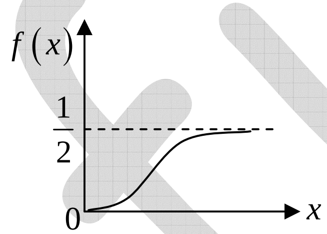
$$= \frac{1}{\pi}\left\{\frac{2\pi\cos n\pi}{n}\right\} \Rightarrow b_n = \begin{cases} \frac{2}{n}; & n = \text{even} \\ \frac{-2}{n}; & n = \text{odd} \end{cases}$$

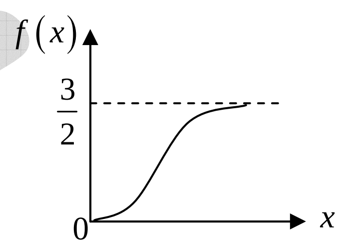
Thus, $b_n \neq 0$

Q46. Which one of the following curves correctly represents (schematically) the solution for the equation $\frac{df}{dx} + 2f = 3 : f(0) = 0$?

(a)


(b)


(c)


(d)


Ans.: (b)

Solution:- $\frac{df}{dx} + 2f = 3;\ f(0) = 0 \Rightarrow \frac{df}{3-2f} = dx \Rightarrow \frac{-1}{2}\ln|3-2f| = x + A$

Since, $f(0) = 0 \Rightarrow A = \frac{-1}{2}\ln|3|$

$$\Rightarrow x = \frac{1}{2}\ln\left|\frac{3}{3-2f}\right| \Rightarrow f = \frac{3}{2}\left(1 - e^{-2x}\right)$$

Now, we can see, at $x = 0,\ f = 0,$ at $x = \infty,\ f = \frac{3}{2}$

Thus option (b) is correct one.

Q47. Consider the transformation to a new set of coordinates (ξ,η) from rectangular Cartesian coordinates (x,y), where $\xi = 2x+3y$ and $\eta = 3x-2y$. In the (ξ,η) coordinate system, the area element $dxdy$ is

(a) $\frac{1}{13}d\xi d\eta$ (b) $\frac{2}{13}d\xi d\eta$ (c) $5d\xi d\eta$ (d) $\frac{3}{5}d\xi d\eta$

Ans.: (a)

Solution:- $$\frac{J(\xi,\eta)}{J(x,y)} = \begin{vmatrix} \frac{\partial \xi}{\partial x} & \frac{\partial \xi}{\partial y} \\ \frac{\partial \eta}{\partial x} & \frac{\partial \eta}{\partial y} \end{vmatrix} = \begin{vmatrix} 2 & 3 \\ 3 & -2 \end{vmatrix} = -13$$

$$\frac{J(x,y)}{J(\xi,\eta)} = \frac{-1}{13} = J$$

Since, area element in $\xi-\eta$ system is, $dA = |J|\,d\xi d\eta = \frac{1}{13}d\xi\, d\eta$

Q48. Let matrix $M = \begin{pmatrix} 4 & x \\ 6 & 9 \end{pmatrix}$. If $\det(M)=0$, then

(a) M is symmetric (b) M is invertible

(c) one eigenvalue is 13 (d) Its eigenvectors are orthogonal

Ans.: (a), (c), (d)

Solution:- Since, $M = \begin{pmatrix} 4 & x \\ 6 & 9 \end{pmatrix}$,

If $|M| = 0 \Rightarrow 36-6x=0 \Rightarrow x=6$

Hence, $M = \begin{pmatrix} 4 & x \\ 6 & 9 \end{pmatrix}$

(a) Here, $M = M^{+}$, so it is symmetric matrix

(b) Determinant $(M)=0$, so noninvertible matrix

(c) For eigenvalue-

$$M - \lambda I = 0 \Rightarrow \begin{vmatrix} 4-\lambda & 6 \\ 6 & 9-\lambda \end{vmatrix} = 0$$

$\Rightarrow (4-\lambda)(9-\lambda)-36=0 \Rightarrow \lambda=0, \lambda=13$

(d) Eigen vectors for distinct eigen values for a symmetric matrix are orthogonal.

Q49. Let $f(x)=3x^6-2x^2-8$. Which of the following statements is (are) true?

(a) The sum of all its roots is zero

(b) The product of its roots is $-\frac{8}{3}$

(c) The sum of all its roots is $\frac{2}{3}$

(d) Complex roots are conjugates of each other.

Ans.: (a), (b), (d)

Solution:- $f(x)=3x^6-2x^2-8$

Now, $3x^6-2x^2-8=0$

$$\Rightarrow x^6-\frac{2}{3}x^2-\frac{8}{3}=0$$

$$\Rightarrow Ax^6+Bx^5+Cx^4+Dx^3+Ex^2+Fx+G=0$$

$$x^6+0.x^5+0.x^4+0.x^3-\frac{2}{3}x^2+0.x-\frac{8}{3}=0$$

Here, sum of roots $=\left(-\frac{B}{A}\right)=0$

And product of roots $=\frac{G}{A}=\left(\frac{-8}{3}\right)$

Since all coefficient are real, then complex roots are conjugate to each other.

Hence, options (a), (b) and (d) are correct.

Q50. The coefficient of x^3 in the Taylor expansion of $\sin(\sin x)$ around $x=0$ is ______.

(Specify your answer upto two digits after the decimal point)

Ans.: 0.33

Solution:- Let $f(x)=\sin(\sin x)$

$$f'(x)=\cos(\sin x).\cos x$$

$$f''(x)=-\sin(\sin x).\cos x.\cos x-\sin x.\cos(\sin x)$$

$$=-\cos^2 x.\sin(\sin x)-\sin x.\cos(\sin x)$$

$$f'''(x) = -\left[-(2\cos x\sin x)\sin(\sin x) + \cos^3 x\cos(\sin x) + \cos x\cos(\sin x) + \sin x(-\sin(\sin x)\cos x)\right]$$

$$= \left[\sin 2x.\sin(\sin x) - \cos^3 x.\cos(\sin x) - \cos x.\cos(\sin x) + \frac{1}{2}\sin 2x.\sin(\sin x)\right]$$

at $x = 0$,

$$f'''(0) = -1 - 1 = -2$$

Hence,

$$f(x) = f(x_0) + \frac{(x - x_1) f'(x_0)}{\underline{|1}} + \frac{(x - x_1)^2 f''(x_0)}{\underline{|2}} + \frac{(x - x_0)^3}{\underline{|3}} f'''(x_1) + ...$$

Hence, coefficient of x^3 is _____

$$= \frac{1}{\underline{|3}}(-2) = \frac{-2}{3 \times 2 \times 1} = \left(-\frac{1}{3}\right) = -0.33$$