Mathematical Physics

JEST-2012

Q1. The value of the integral $\int_0^\infty \frac{\ln x}{\left(x^2+1\right)^2} dx$ is

(a) 0 (b) $\frac{-\pi}{4}$ (c) $\frac{-\pi}{2}$ (d) $\frac{\pi}{2}$

Ans. : (b)

Solution: $\int\limits_0^\infty \frac{\ln x}{\left(x^2+1\right)^2} dx = \int\limits_0^\infty \frac{\ln z}{\left(z^2+1\right)^2} dz$

Let us consider new function $f(z) = \left(\frac{\ln z}{z^2+1}\right)^2$, then $I = \int\limits_0^\infty \left(\frac{\ln z}{z^2+1}\right)^2 dz$

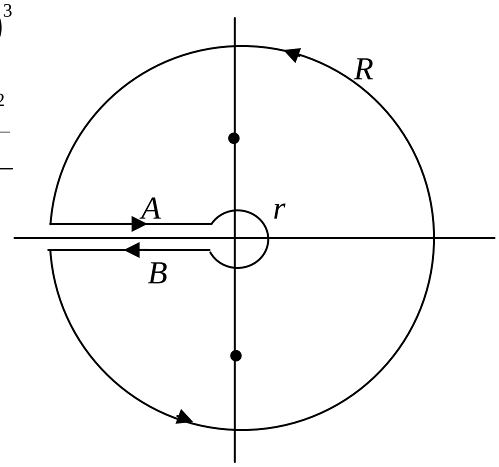
Pole at $z = \pm i$ is simple pole of second order.

Residue at $z = i$ is

$$= \frac{d}{dz}(z-i)^2 \frac{(\ln z)^2}{(z-i)^2 (z+i)^2} = \frac{d}{dz} \frac{(\ln z)^2}{(z+i)^2}$$

$$= \frac{(z+i)^2 2(\ln z).\frac{1}{z} - (\ln z)^2 .2(z+i)}{(z+i)^4} = \frac{(z+i)2\ln(z)\frac{1}{z} - (\ln z)^2 .2}{(z+i)^3}$$

$$= \frac{(2i)2\times\frac{1}{i}\ln i - (\ln i)^2 \cdot 2}{(2i)^3} = \frac{4\frac{i\pi}{2} - \left(\frac{i\pi}{2}\right)^2 \times 2}{-8i} = \frac{2\pi i + \frac{\pi^2}{2}}{-8i}$$



$$\Rightarrow \text{Res}\big|_{z=i} = \frac{-\pi}{4} + \frac{\pi^2}{16} i$$

Similarly, at $z = -i$; $\text{Res}\big|_{z=-i} = \frac{-\pi}{4} - \frac{\pi^2}{16} i$

$$I = \int\limits_0^\infty \left(\frac{\ln z}{z^2+1}\right)^2 dz = 2\pi i\left(\frac{-\pi}{4} + \frac{\pi^2}{16} i - \frac{\pi}{4} - \frac{\pi^2}{16} i\right) = -\pi^2 i$$

$$-\pi^2 i = \left(\int\limits_R \int\limits_A \int\limits_B \int\limits_r\right) f(z)\, dz = \left(\int\limits_A \int\limits_B\right) f(z)\, dz; \qquad \left[\because \int\limits_A \int\limits_B \quad \text{vanish}\right]$$

Along path A; $z=-x+i\varepsilon$ and along path B; $z=-x-i\varepsilon$

Thus $-\pi^2 i=\left(\int_A\int_B\right)f(z)\,dz=-\int_{-\infty}^{0}\left[\frac{\ln(-x+i\varepsilon)}{(-x+i\varepsilon)^2+1}\right]dx-\int_{0}^{\infty}\left[\frac{\ln(-x-i\varepsilon)}{(-x-i\varepsilon)^2+1}\right]dx$

$$\Rightarrow -\pi^2 i=\int_0^\infty\left[\frac{\ln(-x+i\varepsilon)}{(-x+i\varepsilon)^2+1}\right]^2dx-\int_0^\infty\left[\frac{\ln(-x-i\varepsilon)}{(-x-i\varepsilon)^2+1}\right]^2dx$$

$$\Rightarrow -\pi^2 i=\int_0^\infty\left[\frac{\ln(x)+i\pi}{1+x^2}\right]^2dx-\int_0^\infty\left[\frac{\ln(x)-i\pi}{1+x^2}\right]^2dx;\quad \varepsilon\to 0$$

$$\Rightarrow -\pi^2 i=\int_0^\infty\frac{(\ln(x)+i\pi)^2-(\ln(x)-i\pi)^2}{(1+x^2)^2}dx=4\pi i\int_0^\infty\frac{\ln x}{(x^2+1)^2}\Rightarrow\int_0^\infty\frac{\ln x}{(x^2+1)^2}=\frac{-i\pi^2}{4\pi i}=\frac{-\pi}{4}$$

Q2. If [x] denotes the greatest integer not exceeding x, then $\int_0^\infty [x]e^{-x}dx$

(a) $\frac{1}{e-1}$ (b) 1 (c) $\frac{e-1}{e}$ (d) $\frac{e}{e^2-1}$

Ans.: (a)

Solution: $[x]$

$0\le x<1=[x]=0$, $1\le x<2=[x]=1$, $2\le x<3=[x]=2$

Now, $\int_0^\infty [x]e^{-x}dx=\int_0^1[x]e^{-x}dx+\int_1^2[x]e^{-x}dx+\int_2^3[x]e^{-x}dx+\int_3^4[x]e^{-x}dx$

$$\Rightarrow 0+\int_1^2 1.e^{-x}dx+\int_2^3 2.e^{-x}dx+\int_3^4 3.e^{-x}dx=\left[-e^{-x}\right]_1^2+2\left(-e^{-x}\right)_2^3+3\left(-e^{-x}\right)_3^4+....$$

$$=e^{-1}-e^{-2}+2e^{-2}-2e^{-3}+3e^{-3}-3e^{-4}+4e^{-4}-4e^{-5}+$$

$$=e^{-1}+e^{-2}+e^{-3}+e^{-4}+.....\infty$$

$$=\frac{e^{-1}}{1-e^{-1}}=\frac{1}{e-1}\qquad\left(\because\ r=\frac{e^{-2}}{e^{-1}}=e^{-2+1}=e^{-1}\right)$$

Q3. As $x \to 1$, the infinite series $x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + ...$

(a) diverges

(b) converges to unity

(c) converges to $\frac{\pi}{4}$

(d) none of the above

Ans.: (c)

Solution: $\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + ... \Rightarrow \tan^{-1} 1 = \frac{\pi}{4}$

Q4. What is the value of the following series?

$$\left(1 + \frac{1}{2!} + \frac{1}{4!} + ...\right)^2 - \left(1 + \frac{1}{3!} + \frac{1}{5!} + ...\right)^2$$

(a) 0 (b) e (c) e^2 (d) 1

Ans.: (d)

Solution: $e^1 = 1 + 1 + \frac{1^2}{2!} + \frac{1^3}{3!} + - ,\ e^{-1} = 1 - 1 + \frac{1^2}{2!} - \frac{1^3}{3!} .....$

$$\cosh 1 = \frac{e^1 + e^{-1}}{2} = 1 + \frac{1}{2!} + \frac{1}{4!} + ...$$

$$\sinh 1 = \frac{\left(e^1 - e^{-1}\right)}{2} = 1 + \frac{1}{3!} + \frac{1}{5!} + ...$$

i.e., $\cosh^2 1 - \sinh^2 1 = 1$

Q5. An unbiased die is cast twice. The probability that the positive difference (bigger - smaller) between the two numbers is 2 is

(a) $\frac{1}{9}$ (b) $\frac{2}{9}$ (c) $\frac{1}{6}$ (d) $\frac{1}{3}$

Ans.: (b)

Solution: $p(2) = \frac{n(E)}{n(S)}$

The number of ways to come positive difference

$$\left[(3,\ 1),\ (4,\ 2),\ (5,\ 3),(6,\ 4),(1,3)(2,4),(3,5)(4,6)\right]$$

$$p(2) = \frac{8}{36} = \frac{2}{9}$$

Q6. For an $N \times N$ matrix consisting of all ones,

(a) all eigenvalues $= 1$

(b) all eigenvalues $= 0$

(c) the eigenvalues are $1, 2, ...N$

(d) one eigenvalue $= N$, the others $= 0$

Ans.: (d)

Solution: $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = 0,\ \ 2$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} = 0, 0, 3$$

So, far $N \times N$ matrix one eigen value is N and all other eigen values are zero.

JEST-2013

Q7. A box contains 100 coins out of which 99 are fair coins and 1 is a double-headed coin. Suppose you choose a coin at random and toss it 3 times. It turns out that the results of all 3 tosses are heads. What is the probability that the coin you have drawn is the double-headed one?

(a) 0.99

(b) 0.925

(c) 0.75

(d) 0.01

Ans.: (c)

Q8. Compute $\lim_{z \to 0} \frac{\operatorname{Re}(z^2) + \operatorname{Im}(z^2)}{z^2}$

(a) The limit does not exist

(b) 1

(c) $-i$

(d) -1

Ans.: (a)

Solution: $\lim_{z \to 0} \frac{\operatorname{Re}(z^2) + \operatorname{Im}(z^2)}{z^2} = \lim_{z \to 0} \frac{x^2 - y^2 + 2xy}{x^2 - y^2 + 2ixy} = \lim_{\substack{y=0 \\ x \to 0}} \frac{x^2 - y^2 + 2xy}{x^2 - y^2 + 2ixy} = 1$

$$\lim_{\substack{x=0 \\ y \to 0}} \frac{x^2 - y^2 + 2xy}{x^2 - y^2 + 2ixy} = 1 \text{ and } \lim_{\substack{y=x \\ x \to 0}} \frac{x^2 - y^2 + 2xy}{x^2 - y^2 + 2ixy} = -i$$

Q9. The vector field $xz\hat{i}+y\hat{j}$ in cylindrical polar coordinates is

(a) $\rho\left(z\cos^2\phi+\sin^2\phi\right)\hat{e}_\rho+\rho\sin\phi\cos\phi(1-z)\hat{e}_\phi$

(b) $\rho\left(z\cos^2\phi+\sin^2\phi\right)\hat{e}_\rho+\rho\sin\phi\cos\phi(1+z)\hat{e}_\phi$

(c) $\rho\left(z\sin^2\phi+\cos^2\phi\right)\hat{e}_\rho+\rho\sin\phi\cos\phi(1+z)\hat{e}_\phi$

(d) $\rho\left(z\sin^2\phi+\cos^2\phi\right)\hat{e}_\rho+\rho\sin\phi\cos\phi(1-z)\hat{e}_\phi$

Ans.: (a)

Solution: $\vec{A}=xz\hat{i}+y\hat{j}\Rightarrow A_x=xz,\quad A_y=y,\quad A_z=0$

$$A_\rho=\vec{A}\cdot\hat{e}_\rho=A_x\left(\hat{x}\cdot\hat{e}_\rho\right)+A_y\left(\hat{y}\cdot\hat{e}_\rho\right)+A_z\left(\hat{z}\cdot\hat{e}_\rho\right)$$

$$\Rightarrow A_\rho=\rho\cos\phi z\left(\cos\phi\right)+\rho\sin\phi\left(\sin\phi\right)+0\Rightarrow A_\rho=\left(\rho\cos\phi^2 z+\rho\sin^2\phi\right)\hat{e}_\rho$$

$$A_\phi=\vec{A}\cdot\hat{e}_\phi=A_x\left(\hat{x}\cdot\hat{e}_\phi\right)+A_y\left(\hat{y}\cdot\hat{e}_\phi\right)+A_z\left(\hat{z}\cdot\hat{e}_\phi\right)$$

$$\Rightarrow A_\phi=\rho\cos\phi\left(-\sin\phi\right)z+\rho\sin\phi\cdot\cos\phi\Rightarrow A_\phi=\rho\cos\phi\cdot\sin\phi\left(1-z\right)\hat{e}_\phi$$

$$\vec{A}=A_\rho\hat{e}_\rho+A_\phi\hat{e}_\phi+A_z\hat{e}_z=\rho\left(z\cos^2\phi+\sin^2\phi\right)\hat{e}_\rho+\rho\cos\phi\sin\phi\left(1-z\right)\hat{e}_\phi$$

Q10. There are on average 20 buses per hour at a point, but at random times. The probability that there are no buses in five minutes is closest to

(a) 0.07 (b) 0.60 (c) 0.36 (d) 0.19

Ans.: (d)

Solution: From Poision's distribution function,

$$P(n)=\frac{e^{-\lambda}\lambda^n}{\lfloor n}$$

here, $\lambda=20$ buses per hour

$$\Rightarrow\lambda=\frac{5}{3}\text{ buses in five minutes}$$

Therefore, the probability that there are no buses in five minutes,

$$P(n=0)=\frac{e^{-\frac{5}{3}}\left(\frac{5}{3}\right)^0}{\lfloor 0}=e^{-5/3}=0.1886\approx 0.19$$

Thus, option (d) is correct option.

Q11. Two drunks start out together at the origin, each having equal probability of making a step simultaneously to the left or right along the x axis. The probability that they meet after n steps is

(a) $\frac{1}{4^n}\frac{2n!}{n!^2}$ (b) $\frac{1}{2^n}\frac{2n!}{n!^2}$ (c) $\frac{1}{2^n}2n!$ (d) $\frac{1}{4^n}n!$

Ans.: (a)

Solution: The probability of taking $'r'$ steps out of N steps $= {}^N C_r \left(\frac{1}{2}\right)^r \left(\frac{1}{2}\right)^{N-r}$

Total steps $= N = n + n = 2n$

For taking probability of n steps out of N

$$P = {}^N C_n \left(\frac{1}{2}\right)^n \left(\frac{1}{2}\right)^{N-n} = \frac{N!}{(N-n)!n!}\left(\frac{1}{2}\right)^n \left(\frac{1}{2}\right)^{N-n} = \frac{2n!}{n!n!}\left(\frac{1}{2}\right)^{2n} = \frac{2n!}{(n!)^2 4^n}$$

Q12. What is the value of the following series?

$$\left(1-\frac{1}{2!}+\frac{1}{4!}-....\right)^2+\left(1-\frac{1}{3!}+\frac{1}{5!}-...\right)^2$$

(a) 0 (b) e (c) e^2 (d) 1

Ans.: (d)

Solution: $\cos\theta = 1-\frac{\theta^2}{2!}+\frac{\theta^4}{4!}.....$, $\sin\theta = \theta-\frac{\theta^3}{3!}+\frac{\theta^5}{5!}.....$

$$\Rightarrow \left(1-\frac{1}{2!}+\frac{1}{4!}...\right)^2+\left(1-\frac{1}{3!}+\frac{1}{5!}\right)^2 = \cos^2 1+\sin^2 1 = 1 \qquad \left[\because \sin^2\theta+\cos^2\theta = 1\right]$$

Q13. If the distribution function of x is $f(x) = xe^{-x/\lambda}$ over the interval $0 < x < \infty$, the mean value of x is

(a) λ (b) 2λ (c) $\frac{\lambda}{2}$ (d) 0

Ans.: (b)

Solution: Since, it is distribution function so,

$$\langle x\rangle = \frac{\int_{-\infty}^{\infty} xf(x)dx}{\int_{-\infty}^{\infty} f(x)dx} = \frac{\int_0^{\infty} x.xe^{-\frac{x}{\lambda}}dx}{\int_0^{\infty} xe^{-\frac{x}{\lambda}}dx} = \frac{\int_0^{\infty} x^2 e^{\frac{-x}{\lambda}}dx}{\int_0^{\infty} xe^{\frac{-x}{\lambda}}dx} = 2\lambda$$

JEST-2014

Q14. What are the solutions of $f''(x)-2f'(x)+f(x)=0$?

(a) c_1e^x/x (b) c_1x+c_2/x (c) $c_1xe^x+c_2$ (d) $c_1e^x+c_2xe^x$

Ans.: (d)

Solution: Auxiliary equation is, $D^2-2D+1=0 \Rightarrow (D-1)^2=0 \Rightarrow D=+1,\ +1$

$\because$ Roots are equal, then $f(x)=(c_1+c_2x)e^x \Rightarrow f(x)=c_1e^x+c_2xe^x$

Q15. The value of $\int_{0.2}^{2.2} xe^x dx$ by using the one-segment trapezoidal rule is closed to

(a) 11.672 (b) 11.807 (c) 20.099 (d) 24.119

Ans.: (c)

Solution: $h=2.2-0.2=2 \Rightarrow I=\frac{h}{2}\left[y(2.2)+y(0.2)\right]=20.099$ $\left[\because y=xe^x\right]$

Q16. Given the fundamental constants $\hbar$ (Planck's constant), G (universal gravitation constant) and c (speed of light), which of the following has dimension of length?

(a) $\sqrt{\frac{\hbar G}{c^3}}$ (b) $\sqrt{\frac{\hbar G}{c^5}}$ (c) $\frac{\hbar G}{c^3}$ (d) $\sqrt{\frac{\hbar c}{8\pi G}}$

Ans.: (a)

Solution: $\left[\frac{\left[ML^2T^{-1}\right]\left[M^{-1}L^3T^{-2}\right]}{L^3T^{-3}}\right]^{\frac{1}{2}}=\left[L^2\right]^{\frac{1}{2}}=L$

$\hbar=\left[ML^2T^{-1}\right],\ G=\frac{gr^2}{m}=\left[M^{-1}L^3T^{-2}\right]$

Q17. The Laplace transformation of $e^{-2t}\sin 4t$ is

(a) $\frac{4}{s^2+4s+25}$ (b) $\frac{4}{s^2+4s+20}$

(c) $\frac{4s}{s^2+4s+20}$ (d) $\frac{4s}{2s^2+4s+20}$

Ans.: (b)

Solution: $\because L\left[e^{-at}\sin bt\right]=\frac{b}{(s+a)^2+b^2}$

$\Rightarrow L\left[e^{-2t}\sin 4t\right]=\frac{4}{(s+2)^2+4^2}=\frac{4}{s^2+4s+20}$

Q18. Let us write down the Lagrangian of a system as $L(x,\ddot{x},\dddot{x}) = m x\ddot{x} + kx^2 + cx\dddot{x}$. What is the dimension of c?

(a) MLT^{-3} (b) MT^{-2} (c) MT (d) ML^2T^{-1}

Ans.: (c)

Solution: According to dimension rule same dimension will be added or subtracted then dimension of $Mx\ddot{x} =$ dimension of $Cx\dddot{x}$

$$[ML^2T^{-2}] = [C][L][LT^{-3}]$$

$$[C] = \frac{[ML^2T^{-2}]}{[L^2T^{-3}]} = [MT]$$

Q19. The Dirac delta function $\delta(x)$ satisfies the relation $\int_{-\infty}^{\infty} f(x)\delta(x)dx = f(0)$ for a well behaved function $f(x)$. If x has the dimension of momentum then

(a) $\delta(x)$ has the dimension of momentum

(b) $\delta(x)$ has the dimension of $(\text{momentum})^2$

(c) $\delta(x)$ is dimensionless

(d) $\delta(x)$ has the dimension of $(\text{momentum})^{-1}$

Ans.: (d)

Solution: $\int_{-\infty}^{\infty} f(x)\delta(x)dx = f(0)$

$$f(x)\,\delta(x)\,dx = f(0) \Rightarrow [f(x)]\delta(x)\cdot P = [f(0)] \Rightarrow \delta(x) = [P^{-1}]$$

Since, $[f(x)] = [f(0)]$

If $F(x) = \alpha x + \beta$ is force $[M\,LT^{-2}]$

$F(0) = \beta$ is also $[M\,LT^{-2}]$

Q20. The value of limit

$$\lim_{z\to i} \frac{z^{10}+1}{z^6+1}$$

is equal to

(a) 1 (b) 0 (c) $\frac{-10}{3}$ (d) $\frac{5}{3}$

Ans.: (d)

Solution: $\lim_{z\to i} \frac{z^{10}+1}{z^6+1} = \lim_{z\to i} \frac{10z^9}{6z^5} = \lim_{z\to i} \frac{10z^4}{6} = \frac{10}{6} = \frac{5}{3}$

Q21. The value of integral

$$I = \oint_c \frac{\sin z}{2z - \pi} dz$$

with c a circle $|z| = 2$, is

(a) 0 (b) $2\pi i$ (c) πi (d) $-\pi i$

Ans.: (c)

Solution: $I = \oint_C \frac{\sin z}{2z-\pi}$, for pole $2z - \pi = 0 \Rightarrow z = \frac{\pi}{2}$

Residue at $z = \frac{\pi}{2}$ $\quad \because |z| = 2$, so pole will lie within the contour

$$I = \oint_C \frac{e^{iz}}{2\left(z - \frac{\pi}{2}\right)} = \sum R \times 2\pi i$$

$$\text{Res}\Bigg|_{z=\frac{\pi}{2}} = \frac{\left(z-\frac{\pi}{2}\right) e^{iz}}{2\left(z-\frac{\pi}{2}\right)} = \frac{e^{i\pi/2}}{2} = \frac{i}{2}$$ (taking imaginary part); Residue $= \frac{1}{2}$

Now, $I = \frac{1}{2} \times 2\pi i = \pi i$

JEST-2015

Q22. Given an analytic function $f(z)=\phi(x,y)+i\psi(x,y)$, where $\phi(x,y)=x^2+4x-y^2+2y$.

If C is a constant, which of the following relations is true?

(a) $\psi(x,y)=x^2y+4y+C$

(b) $\psi(x,y)=2xy-2x+C$

(c) $\psi(x,y)=2xy+4y-2x+C$

(d) $\psi(x,y)=x^2y-2x+C$

Ans. : (c)

Solution: $u=\phi(x,y)=x^2+4x-y^2+2y$, $v=\psi$

From C.R. equation, $\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}$, $\Rightarrow \frac{\partial \phi}{\partial x}=\frac{\partial \psi}{\partial y}$, $\frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}\Rightarrow \frac{\partial \phi}{\partial y}=-\frac{\partial \psi}{\partial x}$

Now, $\frac{\partial \phi}{\partial x}=2x+4=\frac{\partial \psi}{\partial y}$

$\Rightarrow \psi=2xy+4y+f(x)$ (i)

and $\frac{\partial \phi}{\partial y}=-2y+2\Rightarrow \frac{\partial \psi}{\partial x}=+2y-2$

$\psi=2xy+2x+f(y)$ (ii)

From (i) and (ii), $2xy+4y+f(x)=2xy-2x+f(y)$

$f(x)=-2x,\quad f(y)=4y$

$\psi=2xy+4y-2x+c$

Q23. If two ideal dice are rolled once, what is the probability of getting atleast one '6'?

(a) $\frac{11}{36}$

(b) $\frac{1}{36}$

(c) $\frac{10}{36}$

(d) $\frac{5}{36}$

Ans: (a)

Solution: Number of point in sample space $n(S)=11$

$[(1,6),(2,6),(3,6),(4,6),(5,6),(6,1),(6,2),(6,3),(6,4),(6,5),(6,6)]$

Number of point in population $n(P)=6^2=36$

Probability of getting atleast one '6' on face of dice $=\frac{n(S)}{n(P)}=\frac{11}{36}$

Q24. What is the maximum number of extrema of the function $f(x)=P_k(x)e^{-\left(\frac{x^4}{4}+\frac{x^2}{2}\right)}$, where $x\in(-\infty,\infty)$ and $P_k(x)$ is an arbitrary polynomial of degree k?

(a) $k+2$ (b) $k+6$ (c) $k+3$ (d) k

Ans.: (c)

Solution: $f(x)=P_x(x)e^{-\left(\frac{x^4}{4}+\frac{x^2}{2}\right)}$

$$f'(x)=\left[P_x'(x)+P_x(x)(-1)\left(x^3+x\right)\right]e^{-\left(\frac{x^4}{4}+\frac{x^2}{2}\right)}$$

For maximum number of extrema,

$\Rightarrow f'(x)=0\Rightarrow\left[P_x(x)\left(x^3+x\right)-P'(x)\right]=0$ is polynomial of order $k+3$

From the sign scheme maximum number of extrema $=k+3$

Q25. The Bernoulli polynominals $B_n(s)$ are defined by, $\frac{xe^{xs}}{e^x-1}=\sum B_n(s)\frac{x^n}{n!}$. Which one of the following relations is true?

(a) $\frac{xe^{x(1-s)}}{e^x-1}=\sum B_n(s)\frac{x^n}{(n+1)!}$ (b) $\frac{xe^{x(1-s)}}{e^x-1}=\sum B_n(s)(-1)^n\frac{x^n}{(n+1)!}$

(c) $\frac{xe^{x(1-s)}}{e^x-1}=\sum B_n(-s)(-1)^n\frac{x^n}{n!}$ (d) $\frac{xe^{x(1-s)}}{e^x-1}=\sum B_n(s)(-1)^n\frac{x^n}{n!}$

Ans.: (d)

Solution: $\frac{xe^{xS}}{e^x-1}=\sum B_n(s)\frac{x^n}{\lfloor n}$

Put $s=(s-1)$, $\frac{xe^{x(s-1)}}{e^x-1}=\sum B_n(s-1)\frac{x^n}{\lfloor n}$

Since, $B_n(s-1)=(-1)^n B(s)$

$$\Rightarrow\frac{xe^{x(s-1)}}{e^x-1}=\sum B_n(s)(-1)^n\frac{x^n}{\lfloor n}$$

Q26. Consider the differential equation $G'(x)+kG(x)=\delta(x)$; where k is a constant. Which of the following statements is true?

(a) Both $G(x)$ and $G'(x)$ are continuous at $x=0$.

(b) $G(x)$ is continuous at $x=0$ but $G'(x)$ is not.

(c) $G(x)$ is discontinuous at $x=0$.

(d) The continuity properties of $G(x)$ and $G'(x)$ at $x=0$ depends on the value of k.

Ans.: (c)

Q27. The sum $\sum_{m=1}^{99}\frac{1}{\sqrt{m+1}+\sqrt{m}}$ is equal to

(a) 9 (b) $\sqrt{99}-1$ (c) $\frac{1}{(\sqrt{99}-1)}$ (d) 11

Ans.: (a)

Solution: $\sum_{m=1}^{99}\frac{1}{\sqrt{m+1}+\sqrt{m}}=\sum_{m=1}^{99}\frac{\sqrt{m+1}-\sqrt{m}}{(m+1)-m}=\sum_{m=1}^{99}\sqrt{m+1}-\sqrt{m}$

$=\sqrt{2}-\sqrt{1}+\sqrt{3}-\sqrt{2}......+\sqrt{100}-\sqrt{99}=\sqrt{100}-\sqrt{1}=10-1=9$

JEST-2016

Q28. Given the condition $\nabla^2\phi = 0$, the solution of the equation $\nabla^2\psi = k\vec{\nabla}\phi.\vec{\nabla}\phi$ is given by

(a) $\psi = \frac{k\phi^2}{2}$ (b) $\psi = k\phi^2$ (c) $\psi = \frac{k\phi\ln\phi}{2}$ (d) $\psi = \frac{k\phi\ln\phi}{2}$

Ans.: (a)

Solution: $\nabla^2\phi = 0 \Rightarrow \vec{\nabla}.\left(\vec{\nabla}\phi\right) = 0 \Rightarrow \vec{\nabla}.\left(\vec{\nabla}\phi\right) = 0 \Rightarrow \vec{\nabla}\phi = \alpha\hat{x} + \beta\hat{y} + \gamma\hat{z} \Rightarrow \phi = \alpha x + \beta y + \gamma z$

$$k\vec{\nabla}\phi.\vec{\nabla}\phi = k\left(\alpha^2 + \beta^2 + \gamma^2\right)$$

If $\psi = \frac{k\phi^2}{2} = \frac{k}{2}(\alpha x + \beta y + \gamma z)^2$

$$\Rightarrow \nabla^2\psi = \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2} = k\left(\alpha^2 + \beta^2 + \gamma^2\right) \Rightarrow \nabla^2\psi = k\vec{\nabla}\phi.\vec{\nabla}\phi$$

Q29. The mean value of random variable x with probability density

$p(x) = \frac{1}{\sigma\sqrt{2\pi}}.\exp\left[-\frac{\left(x^2 + \mu x\right)}{\left(2\sigma^2\right)}\right]$ is:

(a) 0 (b) $\frac{\mu}{2}$ (c) $\frac{-\mu}{2}$ (d) σ

Ans. : (a)

Solution: $\langle x\rangle = \frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{\infty} x\exp\left(-\frac{x^2}{2\sigma^2}\right)dx \int_{-\infty}^{\infty}\exp\left(-\frac{\mu x}{2\sigma^2}\right)dx = 0$ (due to odd function)

Q30. Given a matrix $M = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$, which of the following represents $\cos\left(\frac{\pi M}{6}\right)$

(a) $\frac{1}{2}\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$ (b) $\frac{\sqrt{3}}{4}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$ (c) $\frac{\sqrt{3}}{4}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ (d) $\frac{1}{2}\begin{pmatrix} 1 & \sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix}$

Ans. : (b)

Solution: Given, $M = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$

For eigen value, $\begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} = 0 \Rightarrow \lambda = 1, 3$

Now, for $\lambda = 1$,

$$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 1\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$\Rightarrow 2x_1 + x_2 = x_1 \Rightarrow x_1 = -x_2$

Therefore, eigen vector associated with eigen value,

$$\lambda = 1 \text{ is } \phi_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Similarly, for $\lambda = 3$, we get associated eigen vector as,

$$\phi_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Thus, $M = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$

$$\cos\frac{\pi}{6}M = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} \cos\frac{\pi}{6} & 0 \\ 0 & \cos\frac{\pi}{2} \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} \frac{\sqrt{3}}{2} & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{3}}{4} & -\frac{\sqrt{3}}{4} \\ -\frac{\sqrt{3}}{4} & \frac{\sqrt{3}}{4} \end{bmatrix}$$

Therefore, $\cos\left(\frac{\pi M}{6}\right) = \begin{bmatrix} \frac{\sqrt{3}}{4} & -\frac{\sqrt{3}}{4} \\ -\frac{\sqrt{3}}{4} & \frac{\sqrt{3}}{4} \end{bmatrix} = \frac{\sqrt{3}}{4}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$

Thus, option (b) is correct option.

Q31. The sum of the infinite series $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots$ is

(a) 2π (b) π (c) $\frac{\pi}{2}$ (d) $\frac{\pi}{4}$

Ans. : (d)

Solution: The series expansion of $\tan^{-1} x$ in interval $-1 < x \le 1$ is,

$$\tan^{-1} x = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \cdots$$

Putting $x = 1$, we get,

$$\tan^{-1} 1 = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots \Rightarrow 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} \cdots = \frac{\pi}{4}$$

Thus, option (d) is correct.

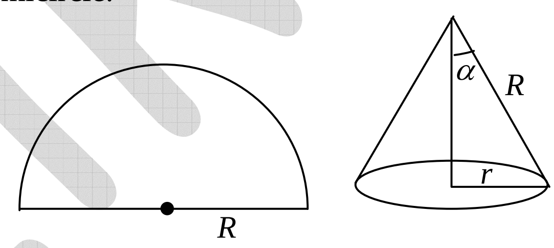
Q32. A semicircular piece of paper is folded to make a cone with the centre of the semicircle as the apex. The half-angle of the resulting cone would be:

(a) 90° (b) 60° (c) 45° (d) 30°

Ans. : (d)

Solution: When the semicircular piece of paper is folded to make a cone, the circumference of base is equal to the circumference of the original semicircle. Let r be the radius of the base of the cone and R be the radius of the semicircle.

Hence, $2\pi r = \pi R \Rightarrow r = \frac{R}{2}$.



The stay height of the come will also be R.

Hence, $\sin\alpha = \frac{R/2}{R} = \frac{1}{2}$

Thus, $\alpha = 30^{0}$

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Q33. $\int_{-\infty}^{+\infty}(x^2+1)\delta(x^2-3x+2)dx = ?$

(a) 1 (b) 2 (c) 5 (d) 7

Ans. : (d)

Solution: $\because\ (x^2-3x+2)=(x^2-x-2x+2)=x(x-1)-2(x-1)=(x-1)(x-2)$

$$\Rightarrow \int_{-\infty}^{\infty}(x^2+1)\delta(x^2-3x+2)dx=\int_{-\infty}^{\infty}(x^2+1)\delta[(x-1)(x-2)]dx$$

$$=\frac{1}{|2-1|}[f(1)+f(2)]=[2+5]=7$$

Q34. Which one is the image of the complex domain $\{z|xy\geq 1, x+y>0\}$ under the mapping $f(z)=z^2$, if $z=x+iy$?

(a) $\{z|xy\geq 1, x+y>0\}$ (b) $\{z|x\geq 2, x+y>0\}$

(c) $\{z|y\geq 2\forall x\}$ (d) $\{z|y\geq 1\forall x\}$

Q35. Let $\Lambda=\begin{pmatrix}1 & 0\\ 0 & 11\end{pmatrix}$ and $M=\begin{pmatrix}10 & 3i\\ -3i & 2\end{pmatrix}$. Similarity, transformation of M to Λ can be performed by

(a) $\frac{1}{\sqrt{10}}\begin{pmatrix}1 & 3i\\ 3i & 1\end{pmatrix}$ (b) $\frac{1}{\sqrt{9}}\begin{pmatrix}1 & -3i\\ 3i & 11\end{pmatrix}$ (c) $\frac{1}{\sqrt{10}}\begin{pmatrix}1 & 3i\\ -3i & 11\end{pmatrix}$ (d) $\frac{1}{\sqrt{9}}\begin{pmatrix}1 & 3i\\ -3i & 1\end{pmatrix}$

Ans. : (a)

Solution: $M=\begin{pmatrix}10 & 3i\\ -3i & 2\end{pmatrix}$

The eigen value of matrix M is 1,11 and corresponding eigen vector are:

$|\phi_1\rangle=\frac{1}{\sqrt{10}}\begin{pmatrix}1\\ 3i\end{pmatrix}, |\phi_2\rangle=\frac{1}{\sqrt{10}}\begin{pmatrix}3i\\ 1\end{pmatrix}$ respectively.

Now, $P=(|\phi_1\rangle\ |\phi_2\rangle)$

$$P=\frac{1}{\sqrt{10}}\begin{pmatrix}1 & 3i\\ 3i & 1\end{pmatrix}$$

Q36. Suppose that we toss two fair coins hundred times each. The probability that the same number of heads occur for both coins at the end of the experiment is

(a) $\left(\frac{1}{4}\right)^{100}\sum_{n=0}^{100}\binom{100}{n}$ (b) $2\left(\frac{1}{4}\right)^{100}\sum_{n=0}^{100}\binom{100}{n}^2$

(c) $\frac{1}{2}\left(\frac{1}{4}\right)^{100}\sum_{n=0}^{100}\binom{100}{n}^2$ (d) $\left(\frac{1}{4}\right)^{100}\sum_{n=0}^{100}\binom{100}{n}^2$

Ans. : (d)

Solution: If we toss one fair coins hundred times, then probability of n number of head occurs at the end of 100 times is

$$^{100}C_n\left(\frac{1}{2}\right)^n\left(\frac{1}{2}\right)^{100-n}$$

Hence, the probability that same number of heads occur for both coins at the end of experiment is

$$\sum_{n=0}^{100}\left({}^{100}C_n\left(\frac{1}{2}\right)^{100}\right)\cdot\left({}^{100}C_n\left(\frac{1}{2}\right)^{100}\right)=\sum_{n=1}^{100}\left({}^{100}C_n\right)^2\left(\frac{1}{2}\right)^{200}=\left(\frac{1}{4}\right)^{100}\sum_{n=1}^{100}\left({}^{100}C_n\right)^2$$

Q37. What is the equation of the plane which is tangent to the surface $xyz=4$ at the point $(1,2,2)$?

(a) $x+2y+4z=12$ (b) $4x+2y+z=12$

(c) $x+4y+z=0$ (d) $2x+y+z=6$

Ans. : (d)

Solution: The surface equation is given by

$$\phi=xyz=4$$

The normal vector to the surface is

$$\vec{n}=\vec{\nabla}\phi=yz\hat{x}+xz\hat{y}+xy\hat{z}$$

At point $(1,2,2)$,

$$\vec{n}=(4\hat{x}+2\hat{y}+2\hat{z}),\ \hat{n}=\frac{\vec{n}}{|\vec{n}|}=\frac{(4\hat{x}+2\hat{y}+2\hat{z})}{\sqrt{16+4+4}}=\frac{(2\hat{x}+\hat{y}+\hat{z})}{\sqrt{6}}$$

The equation of plane at point $(1,2,2)$ is

$$\left[(x-1)\hat{x}+(y-2)\hat{y}+(z-2)\hat{z}\right]\hat{n}=0$$

$$\Rightarrow 2(x-1)+1(y-2)+1(z-2)=0 \Rightarrow 2x+y+z=6$$

Q38. The integral $I=\int_1^{\infty}\frac{\sqrt{x-1}}{(1+x)^2}dx$ is

(a) $\frac{\pi}{\sqrt{2}}$ (b) $\frac{\pi}{2\sqrt{2}}$ (c) $\frac{\sqrt{\pi}}{2}$ (d) $\sqrt{\frac{\pi}{2}}$

Ans. : (b)

Solution: $I=\int_1^{\infty}\frac{\sqrt{x-1}}{(1+x)^2}dx$

Put, $x=(1+z^2)$, $dx=2zdz$

Hence, $I=\int_0^{\infty}\frac{2z^2dz}{(2+z^2)^2}$

Here poles, $(2+z^2)=0 \Rightarrow (z+i\sqrt{2})(z-i\sqrt{2})=0$

Only $(z=i\sqrt{2})$ poles is allowed

Then $R(i\sqrt{2})=\lim_{z\to i\sqrt{2}}\frac{1}{\sqrt{2-1}}\frac{d}{dz}\left[\frac{2z^2(z-i\sqrt{2})^2}{(z-i\sqrt{2})^2(z+i\sqrt{2})^2}\right]$

$$=\lim_{z\to i\sqrt{2}}\left[\frac{(z+i\sqrt{2})^2\cdot 4z-2z^2\cdot 2(z+i\sqrt{2})}{(z+i\sqrt{2})^4}\right]$$

$$=\frac{(2i\sqrt{2})^2\times 4(i\sqrt{2})-2(i\sqrt{2})^2\cdot 2(2i\sqrt{2})}{(2i\sqrt{2})^4}=-\frac{32\sqrt{2}\,i+16\sqrt{2}\,i}{64}=-\frac{16\sqrt{2}\,i}{64}=-\frac{i}{2\sqrt{2}}$$

Hence, $\int_{-\infty}^{\infty}\frac{2z^2}{(2+z^2)^2}dz=2\pi i\left(-\frac{i}{2\sqrt{2}}\right)=\frac{\pi}{\sqrt{2}}$

$$\Rightarrow \int_0^{\infty}\frac{2z^2}{(2+z^2)^2}dz=\frac{\pi}{2\sqrt{2}} \Rightarrow \int_1^{\infty}\frac{\sqrt{x-1}}{(1+x)^2}dx=\frac{\pi}{2\sqrt{2}}$$

Q39. The Fourier transform of the function $\dfrac{1}{x^4+3x^2+2}$ up to proportionality constant is

(a) $\sqrt{2}\exp\left(-k^2\right)-\exp\left(-2k^2\right)$ (b) $\sqrt{2}\exp\left(-|k|\right)-\exp\left(-\sqrt{2}|k|\right)$

(c) $\sqrt{2}\exp\left(-\sqrt{|k|}\right)-\exp\left(-\sqrt{2|k|}\right)$ (d) $\sqrt{2}\exp\left(-\sqrt{2}k^2\right)-\exp\left(-2k^2\right)$

Ans. : (b)

Solution: $f(x)=\dfrac{1}{\left(x^4+3x^2+2\right)}=\dfrac{1}{\left(x^2+1\right)}-\dfrac{1}{\left[x^2+\left(\sqrt{2}\right)^2\right]}$

Now, Fourier transform of $f(x)$ is,

$$F(p)=A\int_{-\infty}^{\infty} f(x)e^{-1kx}dx$$

$$=A\int_{-\infty}^{\infty}\left[\frac{1}{\left(x^2+1\right)}-\frac{1}{x^2+\left(\sqrt{2}\right)^2}\right]e^{-ikx}dx=A\left[\int_{-\infty}^{\infty}\frac{1}{\left(x^2+1\right)}\times e^{-ikx}dx-\int_{-\infty}^{\infty}\frac{e^{-ikx}}{x^2+\left(\sqrt{2}\right)^2}dx\right]$$

$$\because\int_{-\infty}^{\infty}\frac{1}{\left(x^2+a^2\right)}e^{-ikx}dx=\sqrt{\frac{\pi}{2}}\frac{e^{-a|k|}}{a}$$

$$F(k)=A\left[\sqrt{\frac{\pi}{2}}\frac{e^{-|k|}}{1}-\sqrt{\frac{\pi}{2}}\frac{e^{-\sqrt{2}|k|}}{\sqrt{2}}\right]=\frac{A\sqrt{\pi}}{2}\left[\sqrt{2}\exp\left(-|k|\right)-\exp\left(-\sqrt{2}|k|\right)\right]$$

Q40. If $\rho=\dfrac{\left[I+\dfrac{1}{\sqrt{3}}\left(\sigma_x+\sigma_y+\sigma_z\right)\right]}{2}$, where σ's are the Pauli matrices and I is the identity matrix, then the trace of ρ^{2017} is

(a) 2^{2017} (b) 2^{-2017} (c) 1 (d) $\dfrac{1}{2}$

Ans. : (c)

Solution: Given, $\rho=\dfrac{\left[I+\dfrac{1}{\sqrt{3}}\left(\sigma_x+\sigma_y+\sigma_z\right)\right]}{2}$

Now, $\rho^2 = \frac{1}{4}\left[I + \frac{1}{\sqrt{3}}\left(\sigma_x + \sigma_y + \sigma_z\right)\right]\left[I + \frac{1}{\sqrt{3}}\left(\sigma_x + \sigma_y + \sigma_z\right)\right]$

$$= \frac{1}{4}\left[I + \frac{2}{\sqrt{3}}\left(\sigma_x + \sigma_y + \sigma_z\right) + \frac{1}{3}\left(\sigma_x + \sigma_y + \sigma_z\right)^2\right]$$

$$= \frac{1}{4}\left[I + \frac{2}{\sqrt{3}}\left(\sigma_x + \sigma_y + \sigma_z\right) + \frac{1}{3}(3I)\right]$$

$$= \frac{1}{4}\left[2I + \frac{2}{\sqrt{3}}\left(\sigma_x + \sigma_y + \sigma_z\right)\right] = \frac{1}{2}\left[I + \frac{1}{\sqrt{3}}\left(\sigma_x + \sigma_y + \sigma_z\right)\right]$$

$\rho^2 = \rho \Rightarrow \rho^n = \rho.$, where n can be any positive integer

therefore, $\rho^{2017} = \rho$

$$\rho^{2017} = \begin{bmatrix} \frac{1+1/\sqrt{3}}{2} & \frac{1-i}{2\sqrt{3}} \\ \frac{1+i}{2\sqrt{3}} & \frac{1-1/\sqrt{3}}{2} \end{bmatrix}$$

since, Trace of a matrix is equal to sum of their diagonal element, so

Trace of $\rho^{2017} = \frac{1+\frac{1}{\sqrt{3}}}{2} + \frac{1-\frac{1}{\sqrt{3}}}{2} = 1$

Q41. The function $f(x) = \cosh x$ which exists in the range $-\pi \le x \le \pi$ is periodically repeated between $x = (2m-1)\pi$ and $(2m+1)\pi$, where $m = -\infty$ to ∞. Using Fourier series, indicate the correct relation at $x = 0$

(a) $\sum_{n=-\infty}^{\infty} \frac{(-1)^n}{1-n^2} = \frac{1}{2}\left(\frac{\pi}{\cosh \pi} - 1\right)$

(b) $\sum_{n=-\infty}^{\infty} \frac{(-1)^n}{1-n^2} = 2\frac{\pi}{\cosh \pi}$

(c) $\sum_{n=-\infty}^{\infty} \frac{(-1)^{-n}}{1+n^2} = 2\frac{\pi}{\sinh \pi}$

(d) $\sum_{n=1}^{\infty} \frac{(-1)^n}{1+n^2} = \frac{1}{2}\left(\frac{\pi}{\sinh \pi} - 1\right)$

Ans. : (d)

Solution: $f(x) = \cosh x, \ -\pi \le x \le \pi$

Here, $a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi}\cosh x dx = \frac{1}{2\pi}[\sinh x]_{-\pi}^{\pi} = \frac{\sinh \pi}{\pi}$

$b_n = 0$, due to even function

and $a_n = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(e^x + e^{-x}\right)\cos nx dx$ $\quad\left[\because \cosh x = \frac{1}{2}\left(e^x + e^{-x}\right)\right]$

$$a_n = \frac{1}{2\pi}\left[\frac{e^x}{\left(1+n^2\right)}\left(\cos nx + n\sin nx\right) + \frac{e^{-x}}{1+n^2}\left(-\cos nx + n\sin nx\right)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{2\pi}\left[\frac{e^{\pi}(-1)^n}{\left(1+n^2\right)} - \frac{e^{-\pi}(-1)^n}{\left(1+n^2\right)} - \frac{e^{-\pi}(-1)^n}{\left(1+n^2\right)} + \frac{e^{\pi}(-1)^n}{\left(1+n^2\right)}\right]$$

$$= \frac{2(-1)^n\cdot 2\sinh\pi}{2\pi\left(1+n^2\right)} = \frac{2(-1)^n\sinh\pi}{\pi\left(1+n^2\right)}$$

Hence, $f(x) = a_0 + \sum_{n=1}^{\infty}\left(a_n\cos nx + b_n\sin nx\right)$

$$\Rightarrow \cosh x = \frac{\sinh\pi}{\pi} + \sum_{n=1}^{\infty}\frac{2(-1)^n\sinh\pi}{\pi\left(1+n^2\right)}\cos nx$$

At $x = 0$,

$$\sum_{n=1}^{\infty}\frac{2(-1)^n\sinh\pi}{\pi\left(1+n^2\right)} = \left(1 - \frac{\sinh\pi}{\pi}\right) \Rightarrow \sum_{n=1}^{\infty}\frac{(-1)^n}{\left(1+n^2\right)} = \frac{1}{2}\left[\frac{\pi}{\sinh\pi} - 1\right]$$