Classical Mechanics

JEST-2012

Q1. For small angular displacement (i.e., $\sin\theta \approx \theta$), a simple pendulum oscillates harmonically. For larger displacements, the motion

(a) becomes a periodic

(b) remains periodic with the same period

(c) remains periodic with a higher period

(d) remains periodic with a lower period

Ans.: (c)

Q2. A planet orbits a massive star in a highly elliptical orbit, i.e., the total orbital energy E is close to zero. The initial distance of closest approach is R_0. Energy is dissipated through tidal motions until the orbit is circularized with a final radius of R_f. Assume that orbital angular momentum is conserved during the circularization process. then

(a) $R_f = \frac{R_0}{2}$ (b) $R_f = R_0$ (c) $R_f = \sqrt{2}R_0$ (d) $R_f = 2R_0$

Ans. : (d)

Solution: For elliptically motion $E = \frac{1}{2}m\dot{r}^2 + \frac{J^2}{2mr^2} - \frac{GMm}{r}$

$E = 0$ and closest approach is R_0, at $R_0 \Rightarrow \dot{r} = 0$

$$0 = 0 + \frac{J^2}{2mR_0^2} - \frac{GMm}{R_0} \Rightarrow \frac{J^2}{2MR_0^2} = \frac{GMm}{R_0} \Rightarrow J^2 = 2GMm^2R_0$$

From condition of circular orbit

$$\left|\frac{J^2}{mR_f^3}\right| = f(r) = -\frac{\partial V}{\partial r} \Rightarrow \frac{J^2}{mR_f^3} = \frac{GMm}{R_f^2} \Rightarrow \frac{2GMm^2R_0}{mR_f^3} = \frac{GMm}{R_f^2} \Rightarrow R_f = 2R_0$$

Q3. A binary system consists of two stars of equal mass m orbiting each other in a circular orbit under the influence of gravitational forces. The period of the orbit is T. At $t=0$, the motion is stopped and the stars are allowed to fall towards each other. After what time t, expressed in terms of T, do they collide?

$$\int \frac{x^2 dx}{\sqrt{\alpha - x^2}} = \frac{x}{2}\sqrt{\alpha - x^2} + \frac{\alpha}{2}\sin^{-1}\left(\frac{x}{\sqrt{\alpha}}\right)$$

(a) $\sqrt{2}\tau$ (b) $\frac{\tau}{\sqrt{2}}$ (c) $\frac{\tau}{2\sqrt{2}}$ (d) $\frac{\tau}{4\sqrt{2}}$

Ans. : (d)

Solution: $m\frac{d^2x}{dt^2} = -\frac{GMm}{x^2} \Rightarrow \frac{d^2x}{dt^2} = -\frac{GM}{x^2} = -\frac{A}{x^2}$

$$v\frac{dv}{dt} = \frac{-A}{x^2}\frac{dx}{dt} \Rightarrow \frac{d}{dt}\left(\frac{v^2}{2}\right) = \frac{d}{dt}\left(\frac{A}{x}\right) \Rightarrow \frac{v^2}{2} = \frac{A}{x} + C$$

when $x = R$, $v = 0$, then $c = -\frac{A}{R}$

$$\frac{v^2}{2} = \frac{A}{x} - \frac{A}{R} \Rightarrow v = \sqrt{2A}\sqrt{\frac{1}{x} - \frac{1}{R}} \Rightarrow \frac{dx}{dt} = \sqrt{\frac{2A}{R}}\sqrt{\frac{R-x}{x}}$$

$$\int_R^0 \frac{\sqrt{x}}{\sqrt{R-x}}dx = \int_0^t \sqrt{\frac{2A}{R}}dt$$

Put $x = u^2 \Rightarrow dx = 2udu$ and $x = 0, u = 0$ and also, $x = R, u = \sqrt{R}$

$$\int_{\sqrt{R}}^0 \frac{2u^2}{\sqrt{R-u^2}}du = \int_0^t \sqrt{\frac{2A}{R}}dt \Rightarrow -2\left[\frac{u}{2}\sqrt{R-u^2} + \frac{R}{2}\sin^{-1}\frac{u}{\sqrt{R}}\right]_{\sqrt{R}}^0 = \sqrt{\frac{2A}{R}}t$$

$$\Rightarrow +2\left[\frac{\sqrt{R}}{2}\sqrt{R-R} + \frac{R}{2}\sin^{-1}\frac{\sqrt{R}}{\sqrt{R}}\right] = \sqrt{\frac{2A}{R}}t \Rightarrow 2\times\frac{R}{2}\sin^{-1}1 = \sqrt{\frac{2A}{R}}t$$

$$\Rightarrow t\sqrt{\frac{2A}{R}} = 2\times\frac{R}{2}\times\frac{\pi}{2} \Rightarrow t = \frac{R\pi}{2}\times\sqrt{\frac{R}{2A}}$$

$$t = \frac{1}{2\sqrt{2}}\sqrt{\frac{R^3\pi^2}{GM}} \qquad (1)$$

and $\frac{mv^2}{R}=\frac{GMm}{R^2}\Rightarrow v^2=\frac{GM}{R}\Rightarrow v=\frac{2\pi R}{\tau}\Rightarrow\frac{4\pi^2R^2}{\tau^2}=\frac{GM}{R}\Rightarrow\frac{4\pi^2R^3}{GM}=\tau^2$

$$\tau=2\sqrt{\frac{R^3\pi^2}{GM}}\Rightarrow\sqrt{\frac{R^3\pi^2}{GM}}=\frac{\tau}{2} \qquad (2)$$

From (1) and (2), $t=\frac{1}{2\sqrt{2}}\frac{\tau}{2}=\frac{\tau}{4\sqrt{2}}$

Q4. In a certain intertial frame two light pulses are emitted at point $5\,km$ apart and separated in time by $5\,\mu s$. An observer moving at a speed V along the line joining these points notes that the pulses are simultaneous. Therefore V is

(a) $0.7c$ (b) $0.8c$ (c) $0.3c$ (d) $0.9c$

Ans. : (c)

Solution: $\Delta t=0$, $t_2'-t_1'=5\mu s$, $x_2'-x_1'=5km$, $v=-V$

$$t_2-t_1=\frac{t_2'+\left(\frac{-V}{C^2}\right)x_2'}{\sqrt{1-\frac{V^2}{C^2}}}-\frac{t_1'+\left(\frac{-V}{C^2}\right)x_1'}{\sqrt{1-\frac{V^2}{C^2}}}$$

$$\Rightarrow\frac{\left[(t_2'-t_1')-\frac{V}{C^2}(x_2'-x_1')\right]}{\sqrt{1-\frac{V^2}{C^2}}}=0\Rightarrow 5\times10^{-6}-\frac{V}{C^2}\times5\times10^3=0$$

$$\Rightarrow\frac{V}{C^2}=\frac{5\times10^{-6}}{5\times10^3}=10^{-9}\Rightarrow V=3\times10^8\times C\times10^{-9}=0.3c$$

Q5. A jet of gas consists of molecules of mass m, speed v and number density n all moving co-linearly. This jet hits a wall at an angle θ to the normal. The pressure exerted on the wall by the jet assuming elastic collision will be

(a) $p=2mnv^2\cos^2\theta$ (b) $p=2mnv^2\cos\theta$

(c) $p=\sqrt{(3/2)}mnv\cos^2\theta$ (d) $p=mnv^2$

Ans.: (a)

Solution: Change in momentum along y - direction will be cancelled out

$\because$ Change in momentum along $x-$direction, $\Delta p=2mv\cos\theta$

$$\text{Pressure} = \frac{\text{Force}}{\text{Area}} = \frac{\frac{\Delta p}{\Delta t}}{A} = \frac{\Delta p}{A.\Delta t} = \frac{\Delta p}{A\cdot\frac{L}{v\cos\theta}} = \frac{\Delta p v\cos\theta}{A.L}$$

$$\text{Pressure} \quad p' = \frac{2mv\cos\theta\cdot v\cos\theta N}{V}, \ \because \left(n = \frac{N}{V}\right), \left(V = \text{Area}\times L = A\times L\right),$$

$$p' = 2mnv^2\cos^2\theta$$

Q6. If the coordinate q and the momentum p form a canonical pair (q, p), which one of the sets given below also forms a canonical?

(a) $(q, -p)$ (b) (q^2, p^2) (c) $(p, -q)$ (d) $(q^2, -p^2)$

Ans.: (c)

Solution: For canonical pair $(p, -q)$

$$= \frac{\partial p}{\partial q}\cdot\frac{\partial(-q)}{\partial p} - \frac{\partial(p)}{\partial p}\cdot\frac{\partial(-q)}{\partial q} = 0 - (-1) = 1$$

Q7. A girl measures the period of a simple pendulum inside a stationary lift and finds it to be T seconds. If the lift accelerates upward with an acceleration $\frac{g}{4}$, then the time period will be

(a) T (b) $\frac{T}{4}$ (c) $\frac{2T}{\sqrt{5}}$ (d) $2T\sqrt{5}$

Ans.: (c)

Solution: $T = 2\pi\sqrt{\frac{l}{g}}$

Since, lift accelerated upward, then

$$T' = 2\pi\sqrt{\frac{l}{g+g'}} = 2\pi\sqrt{\frac{l}{g+\frac{g}{4}}} = 2\pi\sqrt{\frac{l}{5g}\times 4} = 2\pi\sqrt{\frac{l}{g}}\times\frac{2}{\sqrt{5}} = \frac{2T}{\sqrt{5}}$$

JEST-2013

Q8. In an observer's rest frame, a particle is moving towards the observer with an energy E and momentum P. If c denotes the velocity of light in vacuum, the energy of the particle in another frame moving in the same direction as particle with a constant velocity v is

(a) $\frac{(E+vP)}{\sqrt{1-(v/c)^2}}$ (b) $\frac{(E-vP)}{\sqrt{1-(v/c)^2}}$ (c) $\frac{(E+vP)}{\left[1-(v/c)^2\right]^2}$ (d) $\frac{(E-vP)}{\left[1-(v/c)^2\right]^2}$

Ans.: (a)

Solution: $t' = \frac{t+\frac{vx}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}} \Rightarrow \frac{x'}{c} = \frac{\frac{x}{c}+\frac{v}{c^2}x}{\sqrt{1-\frac{v^2}{c^2}}} \Rightarrow x' = \frac{x+\frac{v}{c}x}{\sqrt{1-\frac{v^2}{c^2}}} \because x = ct,\quad x' = ct'$

Now $x' = E',\quad x = E \Rightarrow E' = \frac{E+\frac{E}{c}v}{\sqrt{1-\frac{v^2}{c^2}}} \Rightarrow E = mc^2,\quad E = Pc \Rightarrow P = \frac{E}{c} \Rightarrow E' = \frac{E+Pv}{\sqrt{1-\frac{v^2}{c^2}}}$

Q9. The free fall time of a test mass on an object of mass M from a height $2R$ to R is

(a) $(\pi/2+1)\sqrt{\frac{R^3}{GM}}$ (b) $\sqrt{\frac{R^3}{GM}}$ (c) $(\pi/2)\sqrt{\frac{R^3}{GM}}$ (d) $\pi\sqrt{\frac{2R^3}{GM}}$

Ans.: (a)

Solution: Equation of motion $\frac{md^2r}{dt^2} = -\frac{GMm}{r^2} \Rightarrow \frac{d^2r}{dt^2} = -\frac{GM}{r^2} \Rightarrow \frac{d^2r}{dt^2} = -\frac{A}{r^2} \qquad \because GM = A$

$$v\frac{dv}{dt} = -\frac{A}{r^2}\frac{dr}{dt} \Rightarrow \frac{d}{dt}\left(\frac{v^2}{2}\right) = \frac{d}{dt}\left(\frac{A}{r}\right) \Rightarrow \frac{v^2}{2} = \frac{A}{r}+C$$

when $r = 2R,\ v = 0$

$$\frac{0}{2} = \frac{A}{2R}+C \Rightarrow C = -\frac{A}{2R} \Rightarrow \frac{v^2}{2} = \frac{A}{r}-\frac{A}{2R} \Rightarrow v = \sqrt{\frac{2A}{r}-\frac{2A}{2R}} \Rightarrow \frac{dr}{dt} = \frac{\sqrt{2A}}{\sqrt{2R}}\sqrt{\frac{2R-r}{r}}$$

$$\int_{2R}^{R}\frac{\sqrt{r}}{\sqrt{2R-r}}dr = -\sqrt{\frac{A}{R}}\int_0^t dt$$

put $r = u^2,\ dr = 2udu$ when $r = 2R,\ u = \sqrt{2R},\ r = R, u = \sqrt{R}$

$$\int_{\sqrt{2R}}^{\sqrt{R}} \frac{u}{\sqrt{2R-u^2}}\times 2udu = -\sqrt{\frac{A}{R}}\int_0^t dt \Rightarrow -\sqrt{\frac{A}{R}}t = 2\int_{\sqrt{2R}}^{\sqrt{R}}\frac{u^2}{\sqrt{2R-u^2}}du$$

$$\Rightarrow -\sqrt{\frac{A}{R}}t = 2\left[-\frac{u}{2}\sqrt{2R-u^2}+\frac{2R}{2}\sin^{-1}\frac{u}{\sqrt{2R}}\right]_{\sqrt{2R}}^{\sqrt{R}}$$

$$\Rightarrow -\sqrt{\frac{A}{R}}t = 2\left[\frac{-\sqrt{R}}{2}\sqrt{2R-R}+\frac{2R}{2}\sin^{-1}\frac{\sqrt{R}}{\sqrt{2R}}+\frac{\sqrt{2R}}{2}\sqrt{2R-2R}-R\sin^{-1}\frac{\sqrt{2R}}{\sqrt{2R}}\right]$$

$$\Rightarrow -\sqrt{\frac{A}{R}}t = 2\left[\frac{-R}{2}+\frac{R\pi}{4}-\frac{R\pi}{2}\right]\Rightarrow t=\frac{R\sqrt{R}}{\sqrt{A}}\left(\frac{\pi}{2}+1\right)\Rightarrow t=\left(\frac{\pi}{2}+1\right)\sqrt{\frac{R^3}{GM}} \quad \because A=GM$$

Q10. Under a Galilean transformation, the coordinates and momenta of any particle or system transform as: $t'=t$, $\vec{r}\,'=\vec{r}+\vec{v}t$ and $\vec{p}\,'=\vec{p}+m\vec{v}$ where $\vec{v}$ is the velocity of the boosted frame with respect to the original frame. A unitary operator carrying out these transformations for a system having total mass M, total momentum $\vec{P}$ and centre of mass coordinate $\vec{X}$ is

(a) $e^{iM\,\vec{v}.\vec{X}/\hbar}e^{it\vec{v}.\vec{P}/\hbar}$ (b) $e^{iM\,\vec{v}.\vec{X}/\hbar}e^{-it\vec{v}.\vec{P}/\hbar}e^{-iMv^2t/(2\hbar)}$

(c) $e^{iM\,\vec{v}.\vec{X}/\hbar}e^{it\vec{v}.\vec{P}/\hbar}e^{iMv^2t/(2\hbar)}$ (d) $e^{it\vec{v}.\vec{P}/\hbar}e^{-iMv^2t/(2\hbar)}$

Ans.: (b)

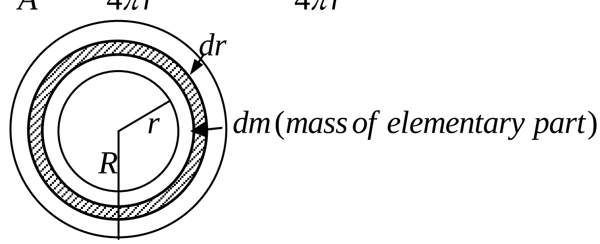
Q11. A spherical planet of radius R has a uniform density ρ and does not rotate. If the planet is made up of some liquid, the pressure at point r from the center is

(a) $\frac{4\pi\rho^2G}{3}\left(R^2-r^2\right)$ (b) $\frac{4\pi\rho G}{3}\left(R^2-r^2\right)$

(c) $\frac{2\pi\rho^2G}{3}\left(R^2-r^2\right)$ (d) $\frac{\rho G}{2}\left(R^2-r^2\right)$

Ans.: (c)

Solution: Pressure $dp=\frac{dm\cdot g}{A}=\frac{dm\cdot g}{4\pi r^2}=\frac{\rho\cdot 4\pi r^2 drGM\frac{r}{R^3}}{4\pi r^2}$



$$\Rightarrow dp = \frac{\rho \cdot 4\pi r^2 drG \cdot \rho \cdot \frac{4\pi}{3} R^3 \frac{r}{R^3}}{4\pi r^2} \Rightarrow dp = \frac{4\pi}{3}\rho^2 Grdr$$

$$\int dp = \int_r^R \frac{4\pi}{3}\rho^2 Grdr \Rightarrow p = \frac{4\pi}{3}\rho^2 G\left(\frac{r^2}{2}\right)_r^R \Rightarrow p = \frac{4\pi}{3}\rho^2 G\left(\frac{R^2}{2} - \frac{r^2}{2}\right)$$

$$\Rightarrow p = \frac{4\pi}{3}\frac{\rho^2 G}{2}\left(R^2 - r^2\right) \Rightarrow p = \frac{2\pi}{3}\rho^2 G\left(R^2 - r^2\right)$$

Q12. A particle of mass m is thrown upward with velocity v and there is retarding air resistance proportional to the square of the velocity with proportionality constant k. If the particle attains a maximum height after time t, and g is the gravitational acceleration, what is the velocity?

(a) $\sqrt{\frac{k}{g}}\tan\left(\sqrt{\frac{g}{k}}t\right)$ (b) $\sqrt{gk}\tan\left(\sqrt{\frac{g}{k}}t\right)$

(c) $\sqrt{\frac{g}{k}}\tan\left(\sqrt{gk}t\right)$ (d) $\sqrt{gk}\tan\left(\sqrt{gk}t\right)$

Ans.: (c)

Solution: Equation of motion $\frac{mdv}{dt} = mg + kv^2 \Rightarrow \frac{dv}{dt} = g + \frac{k}{m}v^2 \Rightarrow \frac{dv}{g + \frac{k}{m}v^2} = dt$

$$\Rightarrow \int \frac{dv}{g + \frac{k}{m}v^2} = \int dt \Rightarrow \int \frac{dv}{\frac{k}{m}\left(\frac{gm}{k} + v^2\right)} = \int dt \Rightarrow \frac{m}{k} \times \frac{1}{\sqrt{\frac{gm}{k}}}\tan^{-1}\frac{v}{\sqrt{\frac{gm}{k}}} = t$$

$$\Rightarrow v = \sqrt{\frac{mg}{k}}\tan\left(\sqrt{\frac{kg}{m}} \cdot t\right)$$

Q13. Consider a uniform distribution of particles with volume density n in a box. The particles have an isotropic velocity distribution with constant magnitude v. The rate at which the particles will be emitted from a hole of area A on one side of this box is

(a) nvA (b) $nv\frac{A}{2}$ (c) $nv\frac{A}{4}$ (d) none of the above

Ans.: (c)

Q14. If, in a Kepler potential, the pericentre distance of particle in a parabolic orbit is r_p while the radius of the circular orbit with the same angular momentum is r_c, then

(a) $r_c = 2r_p$ (b) $r_c = r_p$ (c) $2r_c = r_p$ (d) $r_c = \sqrt{2}r_p$

Ans.: (a)

Solution: Conic equation $\frac{l}{r} = 1 + e\cos\theta$ for parabola $e = 1$ for circle, $e = 0$, $\theta = 0$

$$\frac{l}{r_p} = 1+1,\ \frac{l}{r_C} = 1 \Rightarrow l = 2r_p,\ \ l = r_C \Rightarrow 2r_p = r_C$$

Q15. A K meson (with a rest mass of 494 MeV) at rest decays into a muon (with a rest mass of 106 MeV) and a neutrino. The energy of the neutrino, which can be massless, is approximately

(a) 120 MeV (b) 236 MeV (c) 300 MeV (d) 388 MeV

Ans.: (b)

Solution: $k \to \mu + \nu$, $E_\nu = \frac{\left(m_k^2 - m_\mu^2\right)c^2}{2m_k} = \frac{\left(\frac{494}{c^2}\times\frac{494}{c^2} - \frac{106}{c^2}\times\frac{106}{c^2}\right)c^2}{2\times\frac{494}{c^2}}$

$$\Rightarrow \frac{244036 - 11236}{988} = 235.6275 \approx 236\ \text{MeV}$$

Q16. A light beam is propagating through a block of glass with index of refraction n. If the glass is moving at constant velocity v in the same direction as the beam, the velocity of the light in the glass block as measured by an observer in the laboratory is approximately

(a) $u = \frac{c}{n} + v\left(1 - \frac{1}{n^2}\right)$ (b) $u = \frac{c}{n} - v\left(1 - \frac{1}{n^2}\right)$

(c) $u = \frac{c}{n} + v\left(1 + \frac{1}{n^2}\right)$ (d) $u = \frac{c}{n}$

Ans. : (a)

Solution: now $u = \frac{v + \frac{c}{n}}{1 + \frac{v\cdot c}{c^2\cdot n}} = \left(v + \frac{c}{n}\right)\left(1 + \frac{v}{cn}\right)^{-1} = \left(v + \frac{c}{n}\right)\left(1 - \frac{v}{cn} + \frac{v^2}{c^2n^2}\right)$

$$= v - \frac{v^2}{cn} + \frac{v^3}{c^2n^2} + \frac{c}{n} - \frac{v}{cn^2} + \frac{cv^2}{cn^3} \Rightarrow u = \frac{c}{n} + v\left(1 - \frac{1}{n^2}\right)$$

Q17. The period of a simple pendulum inside a stationary lift is T. If the lift accelerates downwards with an acceleration $\frac{g}{4}$, the period of the pendulum will be

(a) T (b) $\frac{T}{4}$ (c) $\frac{2T}{\sqrt{3}}$ (d) $\frac{2T}{\sqrt{5}}$

Ans.: (c)

Solution: $T = 2\pi\sqrt{\frac{l}{g}} \Rightarrow$ lift accelerates down wards then

$$T' = 2\pi\sqrt{\frac{l}{g-g'}} = 2\pi\sqrt{\frac{l}{g-\frac{g}{4}}} = 2\pi\sqrt{\frac{4l}{3g}} \Rightarrow 2\pi\times 2\sqrt{\frac{l}{3g}} \Rightarrow T' = \frac{2T}{\sqrt{3}}$$

Q18. The velocity of a particle at which the kinetic energy is equal to its rest energy is (in terms of c, the speed of light in vacuum)

(a) $\sqrt{3}c/2$ (b) $3c/4$ (c) $\sqrt{3/5}c$ (d) $c/\sqrt{2}$

Ans.: (a)

Solution: $K.E = mc^2 - m_0c^2$, rest mass energy $= m_0c^2$

$K.E. =$ rest mass energy

$mc^2 - m_0c^2 = m_0c^2 \Rightarrow mc^2 = 2m_0c^2$

$$\frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}}c^2 = 2m_0c^2 \Rightarrow \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} = 2 \Rightarrow 4\left(1-\frac{v^2}{c^2}\right) = 1 \Rightarrow 4\frac{v^2}{c^2} = 3 \Rightarrow v = \frac{\sqrt{3}}{2}c$$

Q19. If the Poisson bracket $\{x, p\} = -1$, then the Poisson bracket $\{x^2 + p, p\}$ is ?

(a) $-2x$ (b) $2x$ (c) 1 (d) -1

Ans.: (a)

Solution: $\{x^2 + p, p\} = \{x^2, p\} + \{p, p\} \Rightarrow x\{x, p\} + \{x, p\}x + 0 \Rightarrow x(-1) + (-1)x \Rightarrow -2x$

Q20. The coordinate transformation $x' = 0.8x + 0.6y,\ \ y' = 0.6x - 0.8y$ represents

(a) a translation (b) a proper rotation

(c) a reflection (d) none of the above

Ans.: (b)

Q21. A small mass M hangs from a thin string and can swing like a pendulum. It is attached above the window of a car. When the car is at rest, the string hangs vertically. The angle made by the string with the vertical when the car has a constant acceleration $a = 1.2\, m/s^2$ is approximately

(a) 1^0 (b) 7^0 (c) 15^0 (d) 90^0

Ans.: (b)

Solution: $T\sin\theta = ma$, $T\cos\theta = mg$, $\tan\theta = \frac{a}{g} \Rightarrow \theta = \tan^{-1}\frac{a}{g} = \tan^{-1}\left(\frac{1.2}{9.8}\right) = 6.98^0 \approx 7^0$

JEST-2014

Q22. A dynamical system with two generalized coordinates q_1 and q_2 has Lagrangian $L = \dot{q}_1^2 + \dot{q}_2^2$. If p_1 and p_2 are the corresponding generalized momenta, the Hamiltonian is given by

(a) $\left(p_1^2 + p_2^2\right)/4$ (b) $\left(\dot{q}_1^2 + \dot{q}_2^2\right)/4$ (c) $\left(p_1^2 + p_2^2\right)/2$ (d) $\left(p_1\dot{q}_1 + p_2\dot{q}_2\right)/4$

Ans.: (a)

Solution: $H = \sum \dot{q}_i p_i - L = \dot{q}_1 p_1 + \dot{q}_2 p_2 - L$

$$\frac{\partial L}{\partial \dot{q}_1} = p_1 = 2\dot{q}_1 \Rightarrow \dot{q}_1 = \frac{p_1}{2} \text{ and } \frac{\partial L}{\partial \dot{q}_2} = p_2 = 2\dot{q}_2 \Rightarrow \dot{q}_2 = \frac{p_2}{2}$$

$$H = \frac{p_1}{2}\cdot p_1 + \frac{p_2}{2}\cdot p_2 - \frac{p_1^2}{4} - \frac{p_2^2}{4} \Rightarrow H = \frac{\left(p_1^2 + p_2^2\right)}{4}$$

Q23. In a certain inertial frame two light pulses are emitted, a distance 5 km apart and separated by $5\mu s$. An observer who is traveling, parallel to the line joining the points where the pulses are emitted, at a velocity v with respect to this frame notes that the pulses are simultaneous. Therefore v is

(a) $0.7c$ (b) $0.8c$ (c) $0.3c$ (d) $0.9c$

Ans.: (c)

Solution: $\left(x_2' - x_1'\right) = 5\times 10^3\ m,\ t_2' - t_1' = 5\times 10^{-6}$ sec

$$\left(t_2 - t_1\right) = \frac{t_2' + \left(\frac{-v}{c^2}x_2'\right)}{\sqrt{1-\frac{v^2}{c^2}}} - \frac{t_1' + \left(\frac{-v}{c^2}x_1'\right)}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{\left[\left(t_2' - t_1'\right) - \frac{v}{c^2}\left(x_2' - x_1'\right)\right]}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$\because t_2 - t_1 = 5\times 10^{-6} - \frac{v}{c^2}5\times 10^3 = 0 \Rightarrow v = 0.3c$$

Q24. A double pendulum consists of two equal masses m suspended by two strings of length l. What is the Lagrangian of this system for oscillations in a plane? Assume the angles θ_1, θ_2 made by the two strings are small (you can use $\cos\theta = 1-\theta^2/2$).

Note: $\omega_0 = \sqrt{g/l}$.

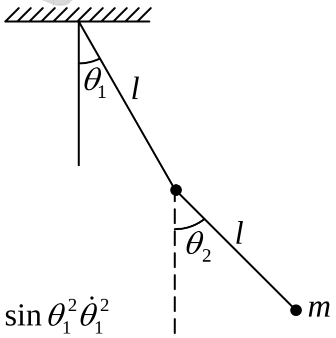
(a) $L \approx ml^2\left(\dot{\theta}_1^2 + \frac{1}{2}\dot{\theta}_2^2 - \omega_0^2\theta_1^2 - \frac{1}{2}\omega_0^2\theta_2^2\right)$

(b) $L \approx ml^2\left(\dot{\theta}_1^2 + \frac{1}{2}\dot{\theta}_2^2 + \dot{\theta}_1\dot{\theta}_2 - \omega_0^2\theta_1^2 - \frac{1}{2}\omega_0^2\theta_2^2\right)$

(c) $L \approx ml^2\left(\dot{\theta}_1^2 + \frac{1}{2}\dot{\theta}_2^2 - \dot{\theta}_1\dot{\theta}_2 - \omega_0^2\theta_1^2 - \frac{1}{2}\omega_0^2\theta_2^2\right)$

(d) $L \approx ml^2\left(\frac{1}{2}\dot{\theta}_1^2 + \frac{1}{2}\dot{\theta}_2^2 + \dot{\theta}_1\dot{\theta}_2 - \omega_0^2\theta_1^2 - \omega_0^2\theta_2^2\right)$

Ans.: (b)



Solution: $x_1 = l\sin\theta_1,\ y_1 = l\cos\theta_1$

$x_2 = x_1 + l\sin\theta_2 \qquad y_2 = y_1 + l\cos\theta_2$

$x_2 = l\sin\theta_1 + l\sin\theta_2,\quad y_2 = l\cos\theta_1 + l\cos\theta_2$

$\dot{x}_2 = l\cos\theta_1\dot{\theta}_1 + l\cos\theta_2\dot{\theta}_2,\ \dot{y}_2 = -l\sin\theta_1\dot{\theta}_1 - l\sin\theta_2\dot{\theta}_2$

$\dot{x}_2^2 + \dot{y}_2^2 = l^2\cos^2\theta_1\dot{\theta}_1^2 + l^2\cos^2\theta_2\dot{\theta}_2^2 + 2l^2\cos\theta_1\dot{\theta}_1\cos\theta_2\dot{\theta}_2 + l^2\sin\theta_1^2\dot{\theta}_1^2$
$+ l^2\sin\theta_2^2\dot{\theta}_2^2 + 2l^2\sin\theta_1\sin\theta_2\dot{\theta}_1\dot{\theta}_2$

$\Rightarrow \dot{x}_2^2 + \dot{y}_2^2 = l^2\dot{\theta}_1^2 + l^2\dot{\theta}_2^2 + 2l^2\cos(\theta_1-\theta_2)\dot{\theta}_1\dot{\theta}_2$ also $\dot{x}_1^2 + \dot{y}_1^2 = l^2\dot{\theta}_1^2$

$$L = T - V = \frac{1}{2}m\left(\dot{x}_1^2 + \dot{y}_1^2 + \dot{x}_2^2 + \dot{y}_2^2\right) - mgy_1 - mgy_2$$

$$\Rightarrow L = \frac{1}{2}m\left(l^2\dot{\theta}_1^2 + l^2\dot{\theta}_1^2 + l^2\dot{\theta}_2^2 + 2l^2\cos(\theta_1-\theta_2)\dot{\theta}_1\dot{\theta}_2\right) + 2mgl\cos\theta_1 + mgl\cos\theta_2$$

$$\Rightarrow L = ml^2\left[\dot{\theta}_1^2 + \frac{1}{2}\dot{\theta}_2^2 + \dot{\theta}_1\dot{\theta}_2 + \frac{2g}{2l}\left(1-\frac{\theta_1^2}{2}\right) + \frac{1}{2}\frac{g}{l}\left(1-\frac{\theta_2^2}{2}\right)\right] \qquad \left[\because \cos(\theta_1-\theta_2)\approx 1\right]$$

$$\Rightarrow L = ml^2\left[\dot{\theta}_1^2 + \frac{1}{2}\dot{\theta}_2^2 + \dot{\theta}_1\dot{\theta}_2 + \frac{g}{l} - \frac{g}{l}\frac{\theta_1^2}{2} + \frac{g}{2l} - \frac{g}{2l}\frac{\theta_2^2}{2}\right]$$

comparing given options, option (b) is correct i.e.

$$L = ml^2\left(\dot{\theta}_1^2 + \frac{1}{2}\dot{\theta}_2^2 + \dot{\theta}_1\dot{\theta}_2 - \frac{\omega_0^2\dot{\theta}_1^2}{2} - \frac{1}{4}\omega_0\dot{\theta}_2^2\right)$$

Q25. A monochromatic wave propagates in a direction making an angle 60° with the x-axis in the reference frame of source. The source moves at speed $v = \frac{4c}{5}$ towards the observer. The direction of the (cosine of angle) wave as seen by the observer is

(a) $\cos\theta' = \frac{13}{14}$ (b) $\cos\theta' = \frac{3}{14}$ (c) $\cos\theta' = \frac{13}{6}$ (d) $\cos\theta' = \frac{1}{2}$

Ans.: (a)

Solution: $v = \frac{4c}{5}$, $u_x' = c\cos 60^\circ = \frac{c}{2}$, $u_y' = c\sin 60^\circ = \frac{\sqrt{3}}{2}c$

Now $u_x = \dfrac{\frac{c}{2} + \frac{4}{5}c}{1 + \frac{c}{2}\cdot\frac{4c}{5c^2}} = \frac{13c}{14} \Rightarrow \cos\theta = \frac{13}{14}$

Q26. The acceleration experienced by the bob of a simple pendulum is

(a) maximum at the extreme positions

(b) maximum at the lowest (central) positions

(c) maximum at a point between the above two positions

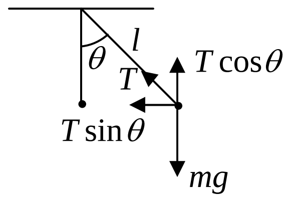
(d) same at all positions

Ans.: (a)

Solution: $T\sin\theta = ma$, $T\cos\theta = mg$

$a = g\tan\theta$ at $\theta = 90^\circ$

a is maximum at extreme position.



Q27. Consider a Hamiltonian system with a potential energy function is given by $V(x) = x^2 - x^4$. Which of the following is correct?

(a) The system has one stable point (b) The system has two stable points

(c) The system has three stable points (d) The system has four stable points

Ans.: (a)

Solution: $V(x)=x^2-x^4$, $\dfrac{\partial V}{\partial x}=2x-4x^3=0 \Rightarrow 2x\left[1-2x^2\right]=0 \Rightarrow x=\pm\dfrac{1}{\sqrt{2}},\ 0$

$$\frac{\partial^2 V}{dx^2}=2-12x^2 \Rightarrow \left.\frac{\partial^2 V}{dx^2}\right|_{x=\pm\frac{1}{\sqrt{2}}}=2-12\times\frac{1}{2}=-4<0$$

For stable point $\dfrac{\partial V}{\partial x}=0$ and $\dfrac{\partial^2 V}{\partial x}>0 \Rightarrow \left.\dfrac{\partial^2 V}{\partial x^2}\right|_{x=0}=2>0$

Q28. Two point objects A and B have masses $1000\,kg$ and $3000\,kg$ respectively. They are initially at rest with a separation equal to $1m$. Their mutual gravitational attraction then draws them together. How far from A's original position will they collide?

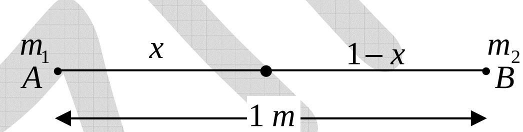
(a) $\frac{1}{3}m$ (b) $\frac{1}{2}m$ (c) $\frac{2}{3}m$ (d) $\frac{3}{4}m$

Ans.: (d)

Solution: Since gravitational force is conservative, therefore they collide at their centre of mass

$m_1x=(1-x)m_2$

$x=3(1-x)\Rightarrow x=\dfrac{3}{4}$



JEST-2015

Q29. The distance of a star from the Earth is 4.25 light years, as measured from the Earth. A space ship travels from Earth to the star at a constant velocity in 4.25 years, according to the clock on the ship. The speed of the space ship in units of the speed of light is,

(a) $\frac{1}{2}$ (b) $\frac{1}{\sqrt{2}}$ (c) $\frac{2}{3}$ (d) $\frac{1}{\sqrt{3}}$

Ans.: (b)

Solution: Proper life-time $\Delta t_0 = \frac{4.25}{c}, \Delta t = \frac{4.25}{v}$

$$\Delta t = \frac{\Delta t_0}{\sqrt{1-v^2/c^2}} \Rightarrow \frac{4.25}{v} = \frac{4.25/c}{\sqrt{1-v^2/c^2}} \Rightarrow \left(\frac{v^2}{c^2}\right) = \left(1-\frac{v^2}{c^2}\right) \Rightarrow v = \frac{1}{\sqrt{2}}c$$

Q30. A classical particle with total energy E moves under the influence of a potential $V(x,y) = 3x^3 + 2x^2y + 2xy^2 + y^3$. The average potential energy, calculated over a long time is equal to,

(a) $\frac{2E}{3}$ (b) $\frac{E}{3}$ (c) $\frac{E}{5}$ (d) $\frac{2E}{5}$

Ans.: (d)

Solution: If one will use virial theorem, then $\langle T\rangle = \frac{n}{2}\langle V\rangle$. If $V \propto r^n$ according to problem $n=3$

So, $\langle E\rangle = \langle T\rangle + \langle V\rangle \Rightarrow \langle E\rangle = \frac{3}{2}\langle V\rangle \Rightarrow \langle V\rangle \Rightarrow \langle V\rangle = \frac{2}{5}\langle E\rangle$

But virial theorem is used only for conservative forces.

Force conservative $\vec{\nabla}\times\vec{F} = 0$, where $\vec{F} = -\vec{\nabla}V$

$\because V(x,y) = 3x^3 + 2x^2y + 2y^2x + y^3 \Rightarrow \vec{\nabla}V = \left(9x^2 + 4xy + 2y^2\right)\hat{i} + \left(2x^2 + 4yx + 3y^2\right)\hat{j}$

$\Rightarrow \vec{\nabla}\times\vec{F} = 0$ i.e., force is conservative in nature.

Therefore, option (d) is correct.

Q31. A chain of mass M and length L is suspended vertically with its lower end touching a weighing scale. The chain is released and falls freely onto the scale. Neglecting the size of the individual links, what is the reading of the scale when a length x of the chain has fallen?

(a) $\frac{Mgx}{L}$ (b) $\frac{2Mgx}{L}$ (c) $\frac{3Mgx}{L}$ (d) $\frac{4Mgx}{L}$

Ans.: (c)

Solution: Reading of scale = impulse + actual weight $=\dfrac{dp}{dt}+\dfrac{Mgx}{L}=\dfrac{d(\Delta mv)}{dt}+\dfrac{Mgx}{L}$

$$\Rightarrow \frac{M}{L}\left(\frac{dx}{dt}\right)v+\frac{Mgx}{L}=\frac{Mv^2}{L}+\frac{Mgx}{L}=\frac{2Mgx}{L}+\frac{Mgx}{L}=\frac{3Mgx}{L}\quad \because v^2=2gx \text{ and } \Delta m=\frac{M}{L}dx$$

Q32. A bike stuntman rides inside a well of frictionless surface given by $z=a\left(x^2+y^2\right)$, under the action of gravity acting in the negative z direction. $\vec{g}=-g\hat{z}$. What speed should be maintain to be able to ride at a constant height z_0 without falling down?

(a) $\sqrt{gz_0}$

(b) $\sqrt{3gz_0}$

(c) $\sqrt{2gz_0}$

(d) The biker will not be able to maintain a constant height, irrespective of speed.

Ans.: (c)

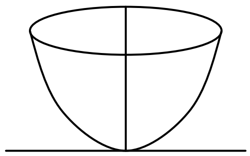
Solution: $z=a\left(x^2+y^2\right)$

Using equation of constrain, we must solve the given system in cylindrical co-ordinate.

$$z=ar^2,\ \dot{z}=2ar\dot{r}\ \Rightarrow L=\frac{1}{2}m\left(\dot{r}^2+r^2\dot{\theta}+\dot{z}^2\right)-mgz$$

$$\Rightarrow L=\frac{1}{2}m\left(\dot{r}^2+r^2\dot{\theta}+4a^2r^2\dot{r}^2\right)-mgar^2=\frac{1}{2}m\left[\dot{r}^2\left(1+4a^2r^2\right)+r^2\dot{\theta}^2\right]-mgar^2$$

Equation of motion



$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right)-\frac{\partial L}{\partial r}=0$$

$$\Rightarrow m\ddot{r}\left(1+4a^2r^2\right)+m\dot{r}^2 4a^2r-mr\dot{\theta}^2+2mgar=0$$

At $z=z_0,\ \dot{r}=0,\quad r=r_0$, so, $mr_0\dot{\theta}^2=2mgar_0$

$$\dot{\theta}^2=2ga\Rightarrow\dot{\theta}=\sqrt{2ga},\ \frac{v}{r_0}=\sqrt{2ga},\ v=\sqrt{2ga}\cdot r_0$$

$$v=\sqrt{2ga}\cdot\left(\frac{z_0}{a}\right)^{1/2}=\sqrt{2gz_0}\qquad\left(\because z_0=ar_0^2\right)$$

Q33. The Lagrangian of a particle is given by $L = \dot{q}^2 - q\dot{q}$. Which of the following statements is true?

(a) This is a free particle

(b) The particle is experiencing velocity dependent damping

(c) The particle is executing simple harmonic motion

(d) The particle is under constant acceleration.

Ans.: (a)

Solution: $\because L = \dot{q}^2 - q\dot{q} \Rightarrow \frac{\partial L}{\partial \dot{q}} = 2\dot{q} - q \Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = 2\ddot{q} - \dot{q}$

$$\because \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = 0$$

$$\Rightarrow 2\ddot{q} - \dot{q} + \dot{q} = 0 \Rightarrow 2\ddot{q} = 0 \Rightarrow \frac{d^2 q}{dt^2} = 0 \Rightarrow \frac{dq}{dt} = C \Rightarrow q = Ct + \alpha$$

Q34. How is your weight affected if the Earth suddenly doubles in radius, mass remaining the same?

(a) Increases by a factor of 4 (b) Increases by a factor of 2

(c) Decreases by a factor of 4 (d) Decreases by a factor of 2

Ans.: (c)

Solution: $W = m\cdot\frac{GM}{R^2}$ and $W' = m\cdot\frac{GM}{(2R)^2} \Rightarrow W' = \frac{W}{4}$

Q35. A spring of force constant k is stretched by x. It takes twice as much work to stretch a second spring by $\frac{x}{2}$. The force constant of the second spring is,

(a) k (b) $2k$ (c) $4k$ (d) $8k$

Ans.: (d)

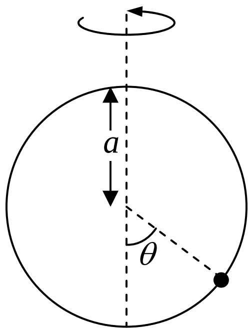
Solution: The relation between energy and maximum displacement is $E = \frac{1}{2}k_1 A^2$

For $A = x$; $E_1 = \frac{1}{2}k_1 x^2$ and for $A = \frac{x}{2}$; $E_1 = \frac{1}{2}k_2\left(\frac{x}{2}\right)^2 = \frac{1}{8}k_2 x^2$

$$\because E_2 = 2E_1 \therefore \frac{1}{8}k_2 x^2 = 2\times\frac{1}{2}k_1 x^2 \Rightarrow k_2 = 8k_1 \Rightarrow k_2 = 8k$$

JEST-2016

Q36. A hoop of radius a rotates with constant angular velocity ω about the vertical axis as shown in the figure. A bead of mass m can slide on the hoop without friction. If $g<\omega^2 a$ at what angle θ apart from 0 and π is the bead stationary (i.e., $\frac{d\theta}{dt}=\frac{d^2\theta}{dt^2}=0$)?



(a) $\tan\theta=\frac{\pi g}{\omega^2 a}$ (b) $\sin\theta=\frac{g}{\omega^2 a}$

(c) $\cos\theta=\frac{g}{\omega^2 a}$ (d) $\tan\theta=\frac{g}{\pi\omega^2 a}$

Ans.: (c)

Solution: The Lagrangian of the system is

$$L=\frac{1}{2}ma^2\left(\dot{\theta}^2+\sin^2\theta\dot{\phi}^2\right)+mga\cos\theta$$

The equation of motion is,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}}\right)-\left(\frac{\partial L}{\partial\theta}\right)=0\Rightarrow ma^2\ddot{\theta}-ma^2\left(\sin\theta\cos\theta\dot{\phi}^2\right)+mga\sin\theta=0$$

When bead is stationary, then

$$\frac{d\theta}{dt}=\frac{d^2\theta}{dt^2}=0\Rightarrow -ma^2\left(\sin\theta\cos\theta\dot{\phi}^2\right)+mga\sin\theta=0,$$

$\Rightarrow\dot{\phi}=\omega$ and $g<\omega^2 a$, then $\cos\theta=\frac{g}{\omega^2 a}$

Q37. The central force which results in the orbit $r=a(1+\cos\theta)$ for a particle is proportional to:

(a) r (b) r^2 (c) r^{-2} (d) None o the above

Ans.: (c)

Solution: $r=a(1+\cos\theta)\Rightarrow u=\frac{1}{r}=\frac{1}{a(1+\cos\theta)}\Rightarrow\frac{du}{d\theta}=\frac{\sin\theta}{a(1+\cos\theta)^2}$

and $\frac{d^2u}{d\theta^2}=2\frac{\sin^2\theta}{a(1+\cos\theta)^3}+\frac{\cos\theta}{a(1+\cos\theta)^2}$

If J is angular momentum and m is mass of particle

$$-\frac{J^2}{m}\left(\frac{d^2u}{d\theta^2}+u\right)=f\left(\frac{1}{u}\right)$$

$$\Rightarrow -\frac{J^2}{m}\left(\frac{d^2u}{d\theta^2}+u\right)=-\frac{J^2}{m}\left(\frac{2\sin^2\theta}{a(1+\cos\theta)^3}+\frac{\cos\theta}{a(1+\cos\theta)^2}+\frac{1}{a(1+\cos\theta)}\right)=f\left(\frac{1}{u}\right)$$

$$\Rightarrow -\frac{J^2}{m}\left(2\frac{1-\cos^2\theta}{a(1+\cos\theta)^3}+\frac{\cos\theta}{a(1+\cos\theta)^2}+\frac{1}{a(1+\cos\theta)}\right)=f\left(\frac{1}{u}\right)$$

Put $u=\frac{1}{a(1+\cos\theta)}$, $\cos\theta=\frac{1-au}{au}$ and solving we get

$f\left(\frac{1}{u}\right)\propto u^2$ so $f(r)\propto r^{-2}$

Q38. Light takes approximately 8 minutes to travel from the Sun to the Earth. Suppose in the frame of the Sun an event occurs at $t=0$ at the Sun and another event occurs on Earth at $t=1$ minute. The velocity of the inertial frame in which both these events are simultaneous is:

(a) $\frac{c}{8}$ with the velocity vector pointing from Earth to Sun

(b) $\frac{c}{8}$ with the velocity vector pointing from Sun to Earth

(c) The events can never be simultaneous - no such frame exists

(d) $c\sqrt{1-\left(\frac{1}{8}\right)^2}$ with velocity vector Pointing from to Earth

Ans.: (a)

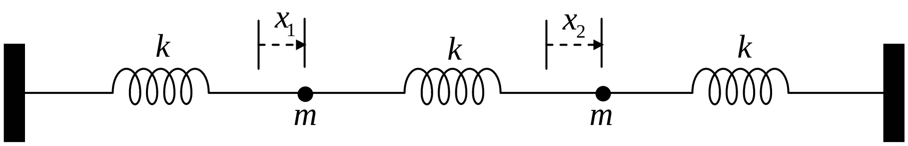
Solution: $x_2'-x_1'=c\times 8\times 60$, $t_2'-t_1'=60$

$$t_2-t_1=0\Rightarrow\frac{t_2'+\frac{vx_2'}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}}-\frac{t_1'+\frac{vx_1'}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}}=0\Rightarrow t_2'-t_1'+\frac{v}{c^2}(x_2'-x_1')=0$$

$$t_2'-t_1'+\frac{v}{c^2}(x_2'-x_1')=0\Rightarrow 60+\frac{v}{c^2}c\times 8\times 60=0\Rightarrow v=-\frac{c}{8}$$

Negative sign indicate frame is moving with the velocity $\frac{c}{8}$ vector pointing from Earth to Sun.

Q39. For the coupled system shown in the figure, the normal coordinates are $x_1 + x_2$ and $x_1 - x_2$ corresponding to the normal frequencies ω_0 and $\sqrt{3}\omega_0$ respectively.



At $t = 0$, the displacements are $x_1 = A$, $x_2 = 0$, and the velocities are $v_1 = v_2 = 0$. The displacement of the second particle at time t is given by:

(a) $x_2(t) = \frac{A}{2}\left(\cos(\omega_0 t) + \cos\left(\sqrt{3}\omega_0 t\right)\right)$ (b) $x_2(t) = \frac{A}{2}\left(\cos(\omega_0 t) - \cos\left(\sqrt{3}\omega_0 t\right)\right)$

(c) $x_2(t) = \frac{A}{2}\left(\sin(\omega_0 t) - \sin\left(\sqrt{3}\omega_0 t\right)\right)$ (d) $x_2(t) = \frac{A}{2}\left(\sin(\omega_0 t) - \frac{1}{\sqrt{3}}\sin\left(\sqrt{3}\omega_0 t\right)\right)$

Ans.: (b)

Solution: Using boundary condition at $t = 0$, $x_2 = 0$ and $v_2 = 0$

Only $x_2(t) = \frac{A}{2}\left(\cos(\omega_0 t) - \cos\left(\sqrt{3}\omega_0 t\right)\right)$ will satisfied

Q40. A cylindrical shell of mass m has an outer radius b and an inner radius a. The moment of inertia of the shell about the axis of the cylinder is:

(a) $\frac{1}{2}m\left(b^2 - a^2\right)$ (b) $\frac{1}{2}m\left(b^2 + a^2\right)$ (c) $m\left(b^2 + a^2\right)$ (d) $m\left(b^2 - a^2\right)$

Ans.: (b)

Solution: $\int_a^b x^2 dm = \frac{m}{\pi\left(b^2 - a^2\right)}\int_a^b x^2 2\pi x dx = \frac{m}{2}\left(b^2 + a^2\right)$

JEST 2017

Q41. A bead of mass M slides along a parabolic wire described by $z = 2\left(x^2 + y^2\right)$. The wire rotates with angular velocity Ω about the z - axis. At what value of Ω does the bead maintain a constant nonzero height under the action of gravity along $-\hat{z}$?

(a) $\sqrt{3g}$ (b) $\sqrt{g}$ (c) $\sqrt{2g}$ (d) $\sqrt{4g}$

Ans. : (d)

Solution: $L = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2 + 16r^2\dot{r}^2\right) - 2mgr^2 \Rightarrow L = \frac{1}{2}m\left(\dot{r}^2\left(1+16r^2\right) + r^2\dot{\theta}^2\right) - 2mgr^2$

The equation of motion is given by

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r} = 0 \Rightarrow m\ddot{r}\left(1+16r^2\right) + 16m\dot{r}^2 r - mr\dot{\theta}^2 + 4mgr = 0$$

At equilibrium, $r = r_0$, $\dot{r} = 0$, $\ddot{r} = 0$

So, $-mr_0\dot{\theta}^2 + 4mgr_0 = 0 \Rightarrow \dot{\theta} = \Omega = \sqrt{4g}$

Q42. $\left(Q_1, Q_2, P_1, P_2\right)$ and $\left(q_1, q_2, p_1, p_2\right)$ are two sets of canonical coordinates, where Q_i and q_i are the coordinates and P_i and p_i are the corresponding conjugate momenta. If $P_1 = q_2$ and $P_2 = p_1$, then which of the following relations is true?

(a) $Q_1 = q_1,\ Q_2 = p_2$ (b) $Q_1 = p_2,\ Q_2 = q_1$

(c) $Q_1 = -p_2,\ Q_2 = q_1$ (d) $Q_1 = q_1,\ Q_2 = -p_2$

Ans. : (c)

Solution: From the symmetry $Q_1 = -p_2,\ Q_2 = q_1$

Q43. $\phi_0(x)$ and $\phi_1(x)$ are respectively are orthonormal wavefunctions of the ground and first excited states of a one dimensional simple harmonic oscillator. Consider the normalised wave function $\psi(x) = c_0\phi_0(x) + c_1\phi_1(x)$, where c_0 and c_1 are real. For what values of c_0 and c_1 will $\langle\psi(x)|x|\psi(x)\rangle$ be maximized?

(a) $c_0 = c_1 = +1/\sqrt{2}$ (b) $c_0 = -c_1 = +1/\sqrt{2}$

(c) $c_0 = +\sqrt{3}/2,\ c_1 = +1/2$ (d) $c_0 = +\sqrt{3}/2,\ c_1 = -1/2$

Ans. : (a)

Solution: $\langle\psi(x)|x|\psi(x)\rangle = 2c_0c_1\langle\phi_0|x|\phi_1\rangle \Rightarrow \left((c_0+c_1)^2-1\right)\langle\phi_0|x|\phi_1\rangle$ $\left[\because\ c_0^2+c_1^2=1\right]$

So, for $\langle\psi(x)|x|\psi(x)\rangle$ to be maximized, $c_0 = c_1 = +1/\sqrt{2}$

Q44. A possible Lagrangian for a free particle is

(a) $L=\dot{q}^2-q^2$ (b) $L=\dot{q}^2-q\dot{q}$ (c) $L=\dot{q}^2-q$ (d) $L=\dot{q}^2-\frac{1}{q}$

Ans. : (b)

Solution: $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right)-\left(\frac{\partial L}{\partial q}\right)=0 \Rightarrow 2\ddot{q}-\dot{q}+\dot{q}=0 \Rightarrow \ddot{q}=0$

Q45. A rod of mass m and length l is suspended from two massless vertical springs with a spring constants k_1 and k_2. What is the Lagrangian for the system, if x_1 and x_2 be the displacements from equilibrium position of the two ends of the rod?

(a) $\frac{m}{8}\left(\dot{x}_1^2+2\dot{x}_1\dot{x}_2+\dot{x}_2^2\right)-\frac{1}{2}k_1x_1^2-\frac{1}{2}k_2x_2^2$

(b) $\frac{m}{2}\left(\dot{x}_1^2+\dot{x}_1\dot{x}_2+\dot{x}_2^2\right)-\frac{1}{4}(k_1+k_2)\left(x_1^2+x_2^2\right)$

(c) $\frac{m}{6}\left(\dot{x}_1^2+x_1\dot{x}_2+\dot{x}_2^2\right)-\frac{1}{2}k_1x_1^2-\frac{1}{2}k_2x_2^2$

(d) $\frac{m}{2}\left(\dot{x}_1^2-2\dot{x}_1\dot{x}_2+\dot{x}_2^2\right)-\frac{1}{4}(k_1-k_2)\left(x_1^2+x_2^2\right)$

Ans. : (c)

Solution: $T=\frac{1}{2}MV_{c.m}^2+\frac{1}{2}I_{c.m}\omega^2=\frac{1}{2}m\left(\frac{\dot{x}_1+\dot{x}_2}{2}\right)^2+\frac{1}{2}\frac{ml^2}{12}\dot{\theta}^2$

Potential energy is, $V=\frac{1}{2}kx_1^2+\frac{1}{2}kx_2^2$

$\sin\theta=\frac{x_2-x_1}{l}$ for small oscillation $\theta=\frac{x_2-x_1}{l}=\dot{\theta}=\frac{\dot{x}_2-\dot{x}_1}{l}$

$$L=\frac{1}{2}m\left(\frac{\dot{x}_1+\dot{x}_2}{2}\right)^2+\frac{1}{2}\frac{ml^2}{12}\left(\frac{\dot{x}_1-\dot{x}_2}{l}\right)^2-\frac{1}{2}kx_1^2-\frac{1}{2}kx_2^2$$

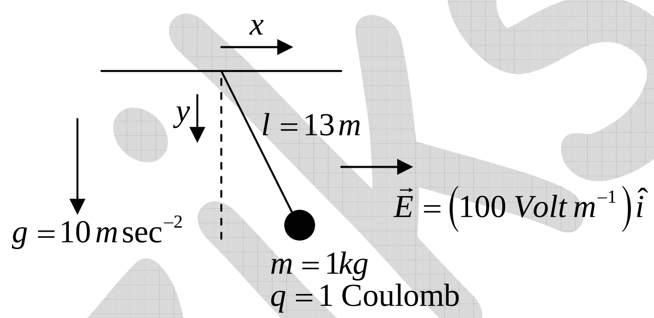
$$=\frac{m}{6}\left(\dot{x}_1^2+x_1\dot{x}_2+\dot{x}_2^2\right)-\frac{1}{2}k_1x_1^2-\frac{1}{2}k_2x_2^2$$

Q46. If the Hamiltonian of a classical particles is $H = \frac{p_x^2 + p_y^2}{2m} + xy$, then $\langle x^2 + xy + y^2 \rangle$ at temperature T is equal to

(a) $k_B T$ (b) $\frac{1}{2} k_B T$ (c) $2k_B T$ (d) $\frac{3}{2} k_B T$

Ans. : (a)

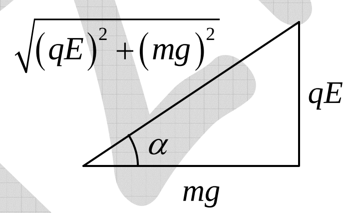
Q47. A simple pendulum has a bob of mass 1 kg and charge 1 Coulomb. It is suspended by a massless string of length 13 m. The time period of small oscillations of this pendulum is T_0. If an electric field $\vec{E} = 100\hat{x} V/m$ is applied, the time period becomes T. What is the value of $(T_0/T)^4$?

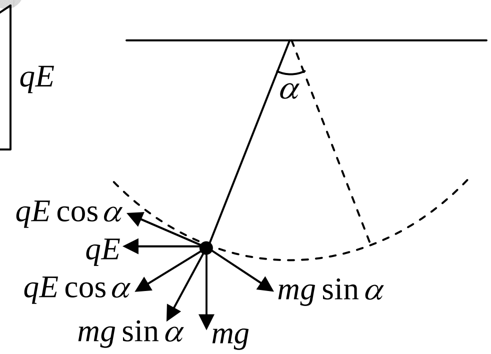


Solution: In equilibrium condition, pendulum is tilted at angle α and is at rest

$\therefore mg \sin\alpha = qE\cos\alpha$

$\tan\alpha = \frac{qE}{mg}$





$\therefore \sin\alpha = \frac{qE}{\sqrt{(2E)^2 + (mg)^2}}$

$\cos\alpha = \frac{mg}{(2E)^2 + (mg)^2}$

When pendulum is displaced by small angle θ the restoring force is

$$F = -[mg\sin(\alpha+\theta) - qE\cos(\alpha+\theta)]$$

$$= -[mg(\sin\alpha\cos\theta + \cos\alpha\sin\theta) - qE(\cos\alpha\cos\theta - \sin\theta\sin\alpha)]$$

$$= -[mg\sin\alpha\cos\theta + mg\cos\alpha\sin\theta - qE\cos\alpha\cos\theta + qE\sin\alpha\sin\theta]$$

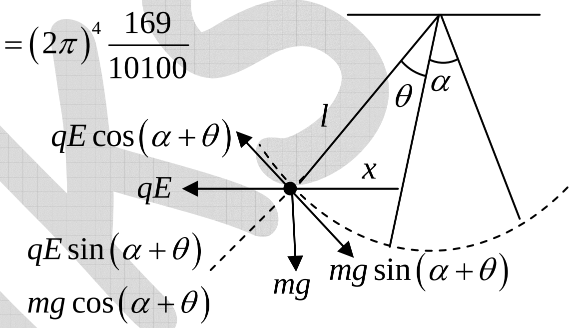
for small angular difference, $\cos\theta \cong 1$ and $\sin\theta = \frac{x}{l}$

$$F = \left[\left(mg\sin\alpha - qE\cos\alpha\right) + mg\cos\alpha.\frac{x}{l} + qE\sin.\frac{x}{l}\right]$$

$$F = -\frac{x}{l}\left[mg.\frac{mg}{\sqrt{(qE)^2+(mg)^2}} + qE\times\frac{qE}{\sqrt{(qE)^2+(mg)^2}}\right] = -\frac{x}{l}.\frac{(mg)^2+(qE)^2}{\sqrt{(2E)^2+(mg)^2}}$$

$$\Rightarrow \frac{d^2x}{dl^2} + \frac{\sqrt{(mg)^2+(qE)^2}}{ml}1 = 0$$

$$\omega^2 = \frac{\sqrt{(mg)^2}}{ml} \Rightarrow T = 2\pi\sqrt{\frac{l}{\sqrt{g^2+\left(\frac{q}{m}E\right)^2}}} \Rightarrow T^4 = (2\pi)^4\frac{169}{10100}$$



As, $T_0 = 2\pi\sqrt{\frac{l}{g}} \Rightarrow T_0^4 = (2\pi)^4\frac{169}{100}$

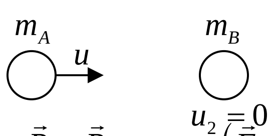
$$\therefore \left(\frac{T_0}{T}\right)^4 = 101$$

Q49. Consider a point particle A of mass m_A colliding elastically with another point particle B of mass m_B at rest, where $\frac{m_B}{m_A} = \gamma$. After collision, the ratio of the kinetic energy of particle B to the initial kinetic energy of particle A is given by

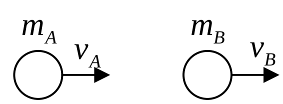
(a) $\frac{4}{\gamma+2+\frac{1}{\gamma}}$ (b) $\frac{2}{\gamma+\frac{1}{\gamma}}$ (c) $\frac{2}{\gamma+2-\frac{1}{\gamma}}$ (d) $\frac{1}{\gamma}$

Ans. : (a)

Solution: Before Collision



After Collision



Since, $\vec{P}_1 = \vec{P}_2$ $\left(\vec{F}_{ext} = 0\right)$

$$\Rightarrow m_A u + 0 = m_B v_B + m_A u_A$$

$$\Rightarrow u = v_A + \gamma v_B$$

Also, $KE_1 = KE_2$

$$\Rightarrow \frac{1}{2}m_A u^2 + 0 = \frac{1}{2}m_A v_A^2 + \frac{1}{2}m_B v_B^2$$

On solving, we get $v_B = \dfrac{2\gamma u}{\gamma + \gamma^2} \Rightarrow \dfrac{v_B}{u} = \dfrac{2}{\gamma + 1}$

$$\frac{KE_B}{KE_A} = \frac{\frac{1}{2}m_B v_B^2}{\frac{1}{2}m_A u^2} = \gamma \times \left(\frac{2}{\gamma+1}\right)^2 = \frac{4\gamma}{\gamma^2 + 2\gamma + 1} = \frac{4}{\gamma + 2 + \frac{1}{\gamma}}$$

Thus, option (a) is correct.

Q50. A toy car is made from a rectangular block of mass M and four disk wheels of mass m and radii r. The car is attached to a vertical wall by a massless horizontal spring with spring constant k and constrained to move perpendicular to the wall. The coefficient of static friction between the wheel of the car and the floor is μ. The maximum amplitude of oscillations of the car above which the wheels start slipping is

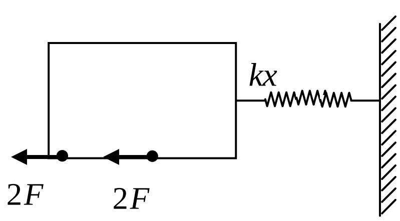
(a) $\dfrac{\mu g(M+2m)(M+4m)}{mk}$ (b) $\dfrac{\mu g\left(M^2 - m^2\right)}{Mk}$

(c) $\dfrac{\mu g(M+m)^2}{2mk}$ (d) $\dfrac{\mu g(M+4m)(M+6m)}{2mk}$

Ans. : (d)

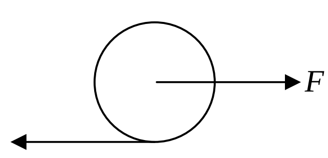
Solution: If F is force on each wheel then

$$kx - 4F = Ma \qquad \text{(i)}$$



For each wheel

$$F_f = \mu\left(mg + \frac{M}{4}g\right)$$



$$F - \mu\left(mg + \frac{M}{4}g\right) = ma$$

When Torque is balanced about bottom most point

$$FR = \left(\frac{3}{2}mR^2\right)\left(\frac{a}{R}\right) = \frac{3}{2}ma$$

$$a = \frac{2\mu\left(mg + \frac{M}{4}g\right)}{m}$$

Putting in equation (i)

$$kx - 6ma = Ma$$

$$kx = (M + 6m)a = \frac{\mu(M+6m)(4m+M)g}{2m}$$

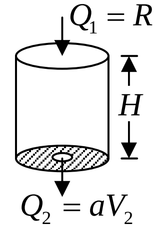
$$x = \frac{\mu(M+6m)(4m+M)g}{2mk}$$

Q51. Water is poured at a rate of $R\,m^3/hour$ from the top into a cylindrical vessel of diameter D. The vessel has a small opening of area $a\left(\sqrt{a} << D\right)$ at the bottom. What should be the minimum height of the vessel so that water does not overflow?

(a) ∞ (b) $\frac{R^2}{2ga^2}$ (c) $\frac{R^2}{2gaD^2}$ (d) $\frac{8R^2}{\pi D^2 g^2}$

Ans. : (b)

Solution: The rate at which liquid coming out of the hole of area 'a' when vessel of height H is filled



$Q_2 = aV_2$, when $V_2 = \sqrt{2gh}$

The rate at which liquid poured in vessel is $Q_1 = R$

$$\therefore\ Q_1 = Q_2 \Rightarrow a\sqrt{2gH} = R \Rightarrow H = \frac{R^2}{2ga^2}$$

Thus, correct option is (b)