

Thermodynamics & Statistical Mechanics

JEST-2012

Q1. A monatomic ideal gas at 170°C is adiabatically compressed to $1/8$ of its original volume. The temperature after compression is

- (a) 2.1°C (b) 17°C (c) -200.5°C (d) 887°C

Ans. : (d)

Solution: $PV^\gamma = \text{constant}$, $PV = RT$

$$\frac{TV^\gamma}{V} = \text{constant}$$

$$\Rightarrow TV^{\gamma-1} = \text{constant}$$

$$\Rightarrow T_1 V_1^{\gamma-1} = T_2 V_2^{\gamma-1} \Rightarrow T_2 = T_1 \left(\frac{V_1}{V_2} \right)^{\gamma-1} \Rightarrow 443 (8)^{\frac{5}{3}-1} = 443 \times (8)^{2/3} = 443 \times 4$$

Temperature in $^\circ\text{C} = 1772 - 273 = 1499$

\therefore Most appropriate answer is option (d)

Q2. Consider a system of particles in three dimensions with momentum \vec{p} and energy $E = c|\vec{p}|$, c being a constant. The system is maintained at inverse temperature β , volume V and chemical potential μ . What is the grand partition function of the system?

- (a) $\exp \left[e^{\beta\mu} 8\pi V / (\beta c h)^3 \right]$ (b) $e^{\beta\mu} 6\pi V / (\beta c h)^2$
 (c) $\exp \left[e^{\beta\mu} 6\pi V / (\beta c h)^3 \right]$ (d) $e^{\beta\mu} 8\pi V / (\beta c h)^2$

Ans. : (a)

Solution: Canonical partition function,

$$z_N = \frac{1}{h^3} \int e^{-\beta H} dp_x dp_y dp_z dx dy dz, \quad E = pc$$

$$z_N = \frac{V}{h^3} \int_0^\infty 4\pi p^2 e^{-E\beta} dp = \frac{V}{h^3} \int_0^\infty 4\pi p^2 e^{-\beta pc} dp = \frac{4\pi V}{h^3} \cdot \frac{\sqrt{3}}{(\beta c)^3} = \frac{8\pi V}{(\beta hc)^3}$$

$$\text{Grand canonical partition function, } z_u = \exp \left[e^{\frac{\mu}{kT}} z_N \right] = \exp \left[e^{\frac{\mu}{kT}} \cdot \frac{8\pi V}{(\beta hc)^3} \right]$$

$$\Rightarrow \exp \left[e^{\beta\mu} \cdot \frac{8\pi V}{(\beta hc)^3} \right]$$

Q3. Consider a system maintained at temperature T , with two available energy states E_1 and E_2 each with degeneracies g_1 and g_2 . If p_1 and p_2 are probabilities of occupancy of the two energy states, what is the entropy of the system?

(a) $S = -k_B [p_1 \ln(p_1 / g_1) + p_2 \ln(p_2 / g_2)]$

(b) $S = -k_B [p_1 \ln(p_1 g_1) + p_2 \ln(p_2 g_2)]$

(c) $S = -k_B [p_1 \ln(p_1^{g_1}) + p_2 \ln(p_2^{g_2})]$

(d) $S = -k_B [(1/p_1) \ln(p_1 / g_1) + (1/p_2) \ln(p_2 / g_2)]$

Ans. : (a)

Solution: $p_i = \frac{\sum g_i e^{-\beta E_i}}{z}$, where z is partition function

$$\Rightarrow \ln p_i = \ln g_i - \beta E_i - \ln z$$

$$\Rightarrow \ln \frac{p_i}{g_i} = -\beta E_i + \frac{F}{kT} \quad [\because F = -kT \ln z]$$

$$\Rightarrow \left\langle \ln \frac{p_i}{g_i} \right\rangle = -\beta \langle E_i \rangle + \beta \langle F \rangle$$

$$\Rightarrow \left\langle \ln \frac{p_i}{g_i} \right\rangle = \beta [F - U] \quad [\because F = U - TS]$$

$$\left\langle \ln \frac{p_i}{g_i} \right\rangle = -\beta \times TS, \quad [\because \beta = \frac{1}{kT}]$$

$$S = -k \left\langle \ln \frac{p_i}{g_i} \right\rangle = -k \left(\sum p_i \ln \frac{p_i}{g_i} \right) = -k \left[p_1 \ln \frac{p_1}{g_1} + p_2 \ln \frac{p_2}{g_2} \right]$$

Q4. Efficiency of a perfectly reversible (Carnot) heat engine operating between absolute temperature T and zero is equal to

(a) 0

(b) 0.5

(c) 0.75

(d) 1

Ans. : (d)

Solution: $\eta = 1 - \frac{T_2}{T_1} = 1 - \frac{0}{T} = 1$

Q5. Consider an ideal gas of mass m at temperature T_1 which is mixed isobarically (i.e. at constant pressure) with an equal mass of same gas at temperature T_2 in a thermally insulated container. What is the change of entropy of the universe?

- (a) $2mC_p \ln\left(\frac{T_1 + T_2}{2\sqrt{T_1 T_2}}\right)$ (b) $2mC_p \ln\left(\frac{T_1 - T_2}{2\sqrt{T_1 T_2}}\right)$
 (c) $2mC_p \ln\left(\frac{T_1 + T_2}{2T_1 T_2}\right)$ (d) $2mC_p \ln\left(\frac{T_1 - T_2}{2\sqrt{T_1 T_2}}\right)$

Ans. : (a)

Solution: Let us consider final temperature will be T

$$mC(T_1 - T) = mC(T - T_2) \Rightarrow T = \frac{T_1 + T_2}{2}$$

$$\Delta S_1 = mC_p \frac{\Delta T}{T}$$

$$\text{Now, } \Delta S = \Delta S_1 + \Delta S_2 \Rightarrow \Delta S = mC_p \int_{T_1}^T \frac{dT}{T} + mC_p \int_{T_2}^T \frac{dT}{T}$$

$$\Rightarrow \Delta S = mC_p \ln\left(\frac{T}{T_1}\right) + mC_p \ln\left(\frac{T}{T_2}\right)$$

$$\Rightarrow \Delta S = 2mC_p \ln \frac{T}{\sqrt{T_1 T_2}} = mC_p \ln \left(\frac{T_1 + T_2}{2\sqrt{T_1 T_2}}\right)^2 \Rightarrow \Delta S = 2mC_p \ln\left(\frac{T_1 + T_2}{2\sqrt{T_1 T_2}}\right)$$

Q6. A collection of N two-level systems with energies 0 and $E > 0$ is in thermal equilibrium at temperature T . For $T \rightarrow \infty$, the specific heat approaches to,

- (a) 0 (b) Nk_B (c) $\frac{3Nk_B}{2}$ (d) ∞

Ans.: (a)

$$\text{Solution: } Z = \sum e^{-\beta E_i} = e^{-\beta \times 0} + e^{-\beta E} \Rightarrow Z = 1 + e^{-\beta E} \Rightarrow \ln z = \ln(1 + e^{-\beta E})$$

$$U = \langle E \rangle = -\frac{\partial}{\partial \beta} \ln z = -\frac{\partial}{\partial \beta} \ln(1 + e^{-\beta E}) = -\frac{1}{1 + e^{-\beta E}} \times e^{-\beta E} (-E) = \frac{E e^{-\beta E}}{1 + e^{-\beta E}}$$

$$\text{Now, } \left(\frac{\partial U}{\partial T} \right)_V = C_V = \frac{\partial}{\partial T} \left(\frac{E e^{-\frac{E}{kT}}}{1 + e^{-\frac{E}{kT}}} \right)$$

$$\Rightarrow C_V = \frac{\left(\frac{E^2}{kT^2} e^{-\frac{E}{kT}} + \frac{E^2}{kT^2} e^{-\frac{2E}{kT}} - \frac{E^2}{kT^2} e^{-\frac{2E}{kT}} \right)}{\left(1 + e^{-\frac{E}{kT}} \right)^2} \Rightarrow C_V = \frac{\frac{E^2}{kT^2} e^{-\frac{E}{kT}}}{\left(1 + e^{-\frac{E}{kT}} \right)^2} \Rightarrow C_V \Big|_{T \rightarrow \infty} = 0$$

Q7. A thermally insulated ideal gas of volume V_1 and temperature T expands to another enclosure of volume V_2 through a porous plug. What is the change in the temperature of the gas?

- (a) 0 (b) $T \ln \left(\frac{V_1}{V_2} \right)$ (c) $T \ln \left(\frac{V_2}{V_1} \right)$ (d) $T \ln \left(\frac{V_2 - V_1}{V_2} \right)$

Ans.: (c)

Solution: $dH = TdS + VdP$, for porous plug Joule Thomson $dH = 0$ and $TdS = 0$ since it is thermally insulated ideal gas

$$VdP = 0$$

$$\therefore VdP = 0 \Rightarrow nRdT = pdV \Rightarrow nRdT = \frac{nRTdV}{V}$$

$$dT = T \frac{dV}{V} \Rightarrow dT = T \int_{V_1}^{V_2} \frac{dV}{V} \Rightarrow dT = T \ln \frac{V_2}{V_1}$$

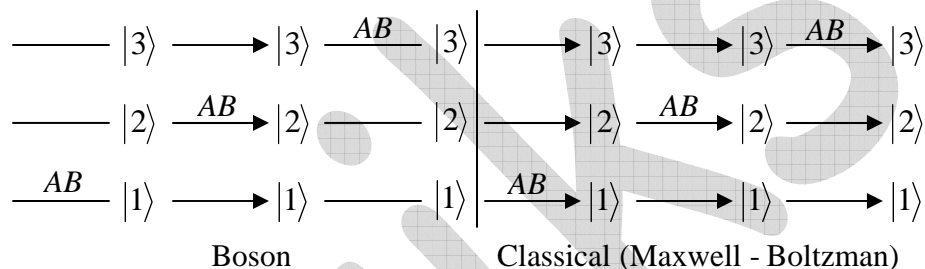
JEST-2013

Q8. Consider a system of two particles A and B . Each particle can occupy one of three possible quantum states $|1\rangle, |2\rangle$ and $|3\rangle$. The ratio of the probability that the two particles are in the same state to the probability that the two particles are in different states is calculated for bosons and classical (Maxwell-Boltzmann) particles. They are respectively

- (a) 1, 0 (b) $\frac{1}{2}, 1$ (c) $1, \frac{1}{2}$ (d) $0, \frac{1}{2}$

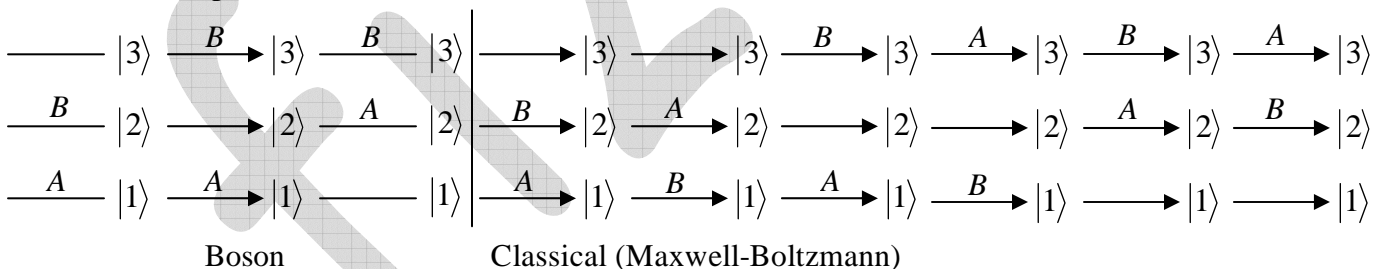
Ans.: (c)

Solution: For two particle in same state:



Probability ratio: $\frac{1/3}{1/3} = 1$

For two particle in different states



Probability ratio: $\frac{1/3}{2/3} = \frac{1}{2}$

Q9. For a diatomic ideal gas near room temperature, what fraction of the heat supplied is available for external work if the gas is expanded at constant pressure?

- (a) $\frac{1}{7}$ (b) $\frac{5}{7}$ (c) $\frac{3}{4}$ (d) $\frac{2}{7}$

Ans.: (d)

Solution: It is isobaric process (constant pressure). Then $\delta Q = nC_p \Delta T \Rightarrow \Delta W = nR\Delta T$

In this process δQ is heat exchange during process.

Function of heat supplied

$$= \frac{\delta W}{\Delta Q} = \frac{nR\Delta T}{nC_p\Delta T} = \frac{R}{R\frac{\gamma}{\gamma-1}} = \frac{\gamma-1}{\gamma} = 1 - \frac{1}{\gamma}$$

$$= 1 - \frac{1}{\left(1 + \frac{2}{f}\right)} \quad \left[\because \gamma = \frac{C_p}{C_v} \Rightarrow C_p = \frac{\gamma R}{\gamma-1} \right]$$

$$= 1 - \frac{f}{f+2} \quad [f = \text{degree of freedom, for diatomic molecule } f = 5]$$

$$\Rightarrow 1 - \frac{5}{5+2} = \frac{2}{7}$$

Q10. A metal bullet comes to rest after hitting its target with a velocity of 80 m/s . If 50% of the heat generated remains in the bullet, what is the increase in its temperature? (The specific heat of the bullet = $160 \text{ Joule / kg / } ^\circ\text{C}$)

- (a) 14°C (b) 12.5°C (c) 10°C (d) 8.2°C

Ans.: (c)

Solution: Conservation of momentum 50% of $\frac{1}{2}mv^2 = mc\Delta T \Rightarrow \frac{1}{2} \frac{80 \times 80}{2} = 160 \Delta T$

$$\Rightarrow \Delta T = \frac{80 \times 80}{4} \times \frac{1}{160} = 10^\circ \text{C}$$

Q11. Consider a particle with three possible spin states: $s = 0$ and ± 1 . There is a magnetic field h present and the energy for a spin state s is $-hs$. The system is at a temperature T . Which of the following statements is true about the entropy $S(T)$?

- (a) $S(T) = \ln 3$ at $T = 0$, and 3 at high T (b) $S(T) = \ln 3$ at $T = 0$, and 0 at high T
 (c) $S(T) = 0$ at $T = 0$, and 3 at high T (d) $S(T) = 0$ at $T = 0$, and $\ln 3$ at high T

Ans.: (d)

Solution: $S = k \ln \omega$, where ω = number of microstates

$$S = k \ln 3 \quad \omega = 3, \text{ at height } T \text{ and at } T = 0, \text{ it is perfect ordered i.e. } S = 0$$

Q14. The temperature of a thin bulb filament (assuming that the resistance of the filament is nearly constant) of radius r and length L is proportional to

- (a) $r^{1/4}L^{-1/2}$ (b) L^2r (c) $L^{1/4}r^{-1}$ (d) r^2L^{-1}

Ans.: (a)

Q15. Ice of density ρ_1 melts at pressure P and absolute temperature T to form water of density ρ_2 . The latent heat of melting of 1 gram of ice is L . What is the change in the internal energy ΔU resulting from the melting of 1 gram of ice?

- (a) $L + P\left(\frac{1}{\rho_2} - \frac{1}{\rho_1}\right)$ (b) $L - P\left(\frac{1}{\rho_2} - \frac{1}{\rho_1}\right)$
 (c) $L - P\left(\frac{1}{\rho_1} - \frac{1}{\rho_2}\right)$ (d) $L + P\left(\frac{1}{\rho_1} - \frac{1}{\rho_2}\right)$

Ans.: (d)

Solution: $dU = dQ - \delta W = dQ - PdV$

$$dU = mL - PdV \Rightarrow dU = L - P \int_{\rho_1}^{\rho_2} \left(-\frac{1}{\rho^2}\right) d\rho = L + P \left[\frac{1}{\rho_1} - \frac{1}{\rho_2}\right]$$

$$\because V = \frac{1}{\rho} \Rightarrow dV = -\frac{1}{\rho^2} d\rho$$

Q16. What is the contribution of the conduction electrons in the molar entropy of a metal with electronic coefficient of specific heat?

- (a) γT (b) γT^2 (c) γT^3 (d) γT^4

Ans.: (a)

Solution: $C_v = BT^3 + AT$

Q17. Consider a system of $2N$ non-interacting spin $1/2$ particles each fixed in position and carrying a magnetic moment μ . The system is immersed in a uniform magnetic field B .

The number of spin up particles for which the entropy of the system will be maximum is

- (a) 0 (b) N (c) $2N$ (d) $N/2$

Ans.: (b)

Solution: Let us consider n number of spin out of $2N$ particle have spin up remaining $2N - n$ is down.

$$\text{Number of ways, } \omega = \begin{cases} 2^N C_n & \text{for spin } \frac{1}{2} \text{ (up)} \\ 2^N C_{2N-n} & \text{for spin } \frac{1}{2} \text{ (down)} \end{cases},$$

$$\text{Entropy, } S = k \ln \omega \Rightarrow S = k \ln 2^N C_{2N-n} + k \ln 2^N C_n$$

$$S = k \left\{ \left[\ln \frac{2N!}{(n!)(2N-n)!} \right] + \left[\ln \frac{2N!}{(n!)(2N-n)!} \right] \right\}$$

$$S = 2k [(\ln 2N! - \ln n! - \ln(2N-n)!)]$$

$$S = 2k [2N \ln 2N - 2N - n \ln n + n - \{(2N-n) \ln(2N-n) - (2N-n)\}]$$

$$[\because \ln N! = N \ln N - N!]$$

$$S = 2k [2N \ln 2N - 2N - n \ln n + n - 2N \ln(2N-n) + n \ln(2N-n) + (2N-n)]$$

$$S = 2k [2N \ln 2N - n \ln n - 2N \ln(2N-n) + n \ln(2N-n)]$$

Now for maximum entropy at equilibrium for spin $\frac{1}{2}$ up particle,

$$\frac{dS}{dn} = 0$$

$$\frac{dS}{dn} = 2k \left[-\frac{n}{n} \cdot 1 - \ln n - \frac{2N}{2N-n} (-1) + \frac{n}{2N-n} (-1) + \ln(2N-n) \right]$$

$$= 2k \left[-1 - \ln n + \frac{2N}{2N-n} - \frac{n}{2N-n} + \ln(2N-n) \right]$$

$$= 2k \left[-1 + \frac{2N-n}{2N-n} + \ln(2N-n) - \ln n \right] \Rightarrow 2k \left[-1 + 1 + \ln \frac{(2N-n)}{n} \right] = 0$$

$$\therefore 2k \neq 0$$

$$\therefore \ln \left(\frac{2N-n}{n} \right) = 0 \Rightarrow \frac{2N-n}{n} = 1 \Rightarrow 2N = 2n \Rightarrow n = N$$

Q18. For which gas the ratio of specific heats (C_p / C_v) will be the largest?

- (a) mono-atomic (b) di-atomic (c) tri-atomic (d) hexa-atomic

Ans.: (a)

Solution: $\frac{C_p}{C_v} = \gamma = \left(1 + \frac{2}{f}\right)$, where f is degree of freedom.

For monoatomic: $f = 3$, For diatomic: $f = 6$, For Triatomic: $f = 9$

For hexaatomic: $f = 18$



JEST-2015

Q19. For a system in thermal equilibrium with a heat bath at temperature T , which one of the following equalities is correct? $\left(\beta = \frac{1}{k_B T}\right)$

(a) $\frac{\partial}{\partial \beta} \langle E \rangle = \langle E \rangle^2 - \langle E^2 \rangle$

(b) $\frac{\partial}{\partial \beta} \langle E \rangle = \langle E^2 \rangle - \langle E \rangle^2$

(c) $\frac{\partial}{\partial \beta} \langle E \rangle = \langle E^2 \rangle + \langle E \rangle^2$

(d) $\frac{\partial}{\partial \beta} \langle E \rangle = -(\langle E^2 \rangle + \langle E \rangle^2)$

Ans.: (a)

Solution: $\therefore \langle E \rangle = \frac{\sum_i E_i e^{-\beta E_i}}{\sum_i e^{-\beta E_i}}$

$$\frac{\partial \langle E \rangle}{\partial \beta} = \frac{\sum_i E_i^2 e^{-\beta E_i}}{\sum_i e^{-\beta E_i}} + \frac{\sum_i E_i^2 e^{-\beta E_i} \cdot e^{-\beta E_i}}{\left(\sum_i e^{-\beta E_i}\right)^2} = \frac{\sum_i E_i^2 e^{-\beta E_i}}{\sum_i e^{-\beta E_i}} + \frac{\sum_i E_i^2 e^{-2\beta E_i}}{\left(\sum_i e^{-\beta E_i}\right)^2}$$

$$\Rightarrow \frac{\partial \langle E \rangle}{\partial \beta} = \langle E \rangle^2 - \langle E^2 \rangle$$

Q20. An ideal gas is compressed adiabatically from an initial volume V to a final volume αV and a work W is done on the system in doing so. The final pressure of the gas will be

$\left(\gamma = \frac{C_P}{C_V}\right)$

(a) $\frac{W}{V^\gamma} \frac{1-\gamma}{\alpha - \alpha^\gamma}$

(b) $\frac{W}{V^\gamma} \frac{\gamma-1}{\alpha - \alpha^\gamma}$

(c) $\frac{W}{V} \frac{1-\gamma}{\alpha - \alpha^\gamma}$

(d) $\frac{W}{V} \frac{\gamma-1}{\alpha - \alpha^\gamma}$

Ans.: (c)

Solution: Work done in adiabatic process,

$$W = \frac{P_2 V_2 - P_1 V_1}{1-\gamma}$$

$$\therefore P_2 V_2^\gamma = P_1 V_1^\gamma \Rightarrow P_1 = P_2 \left(\frac{V_2}{V_1}\right)^\gamma \Rightarrow P_1 = P_2 (\alpha)^\gamma$$

$$W = \frac{P_2 \alpha V - P_2 \alpha^\gamma V}{(1-\gamma)} \Rightarrow P_2 = \frac{W}{V} \frac{(1-\gamma)}{(\alpha - \alpha^\gamma)}$$

Q21. A particle in thermal equilibrium has only 3 possible states with energies $-\epsilon$, 0 , ϵ . If the system is maintained at a temperature, $T \gg \frac{\epsilon}{k_B}$, then the average energy of the particle can be approximated to,

- (a) $\frac{2\epsilon^2}{3k_B T}$ (b) $\frac{-2\epsilon^2}{3k_B T}$ (c) $\frac{-\epsilon^2}{k_B T}$ (d) 0

Ans.: (b)

Solution: $\langle E \rangle = \frac{-\epsilon e^{\frac{\epsilon}{kT}} + 0 + \epsilon e^{-\frac{\epsilon}{kT}}}{e^{\frac{\epsilon}{kT}} + 1 + e^{-\frac{\epsilon}{kT}}} = \epsilon \left(\frac{e^{\frac{\epsilon}{kT}} - e^{-\frac{\epsilon}{kT}}}{1 + e^{\frac{\epsilon}{kT}} + e^{-\frac{\epsilon}{kT}}} \right)$

$$\Rightarrow \langle E \rangle = \frac{\left[\left(1 - \frac{\epsilon}{kT}\right) - \left(1 + \frac{\epsilon}{kT}\right) \right]}{1 + \left(1 - \frac{\epsilon}{kT}\right) + \left(1 + \frac{\epsilon}{kT}\right)} = \frac{-2\epsilon^2}{3kT}$$

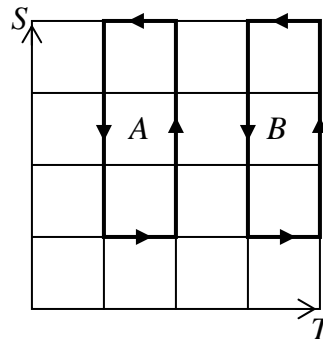
Q22. The blackbody at a temperature of 6000K emits a radiation whose intensity spectrum peaks at 600nm . If the temperature is reduced to 300K , the spectrum will peak at,

- (a) $120\mu\text{m}$ (b) $12\mu\text{m}$ (c) 12mm (d) 120mm

Ans.: (b)

Solution: $\lambda_1 T_1 = \lambda_2 T_2 \Rightarrow \lambda_2 = \frac{\lambda_1 T_1}{T_2} = \frac{600 \times 6000}{300} = 12000\text{ nm} = 12\mu\text{m}$

Q23. The entropy-temperature diagram of two Carnot engines, A and B, are shown in the figure 4. The efficiencies of the engines are η_A and η_B respectively. Which one of the following equalities is correct?



- (a) $\eta_A = \frac{\eta_B}{2}$
 (b) $\eta_A = \eta_B$
 (c) $\eta_A = 3\eta_B$
 (d) $\eta_A = 2\eta_B$

Ans.: (d)

Solution: $\eta = \frac{\Delta W}{Q_1}$, where ΔW = area under the curve, Q_1 = area under high temperature

$$\eta_A = \frac{(2T - T)(3S - 0)}{2T(3S - 0)} = \frac{T}{2T} = \frac{1}{2} \quad \text{and} \quad \eta_B = \frac{(4T - 3T)(3S - 0)}{4T(3S - 0)} = \frac{T}{4T} = \frac{1}{4}$$

$$\Rightarrow \frac{\eta_A}{\eta_B} = \frac{1/2}{1/4} = 2 \Rightarrow \eta_A = 2\eta_B$$

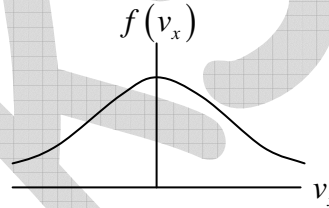
Q24. Electrons of mass m in a thin, long wire at a temperature T follow a one-dimensional Maxwellian velocity distribution. The most probable speed of these electrons is,

- (a) $\sqrt{\left(\frac{kT}{2\pi m}\right)}$ (b) $\sqrt{\left(\frac{2kT}{m}\right)}$ (c) 0 (d) $\sqrt{\left(\frac{8kT}{\pi m}\right)}$.

Ans.: (c)

Solution: $f(v_x) = \left(\frac{m}{2\pi kT}\right)^{1/2} e^{-\frac{mv_x^2}{2kT}} dv_x; \quad -\infty < v_x < \infty$

Most probable speed $v_x = 0$



JEST-2016

Q25. An ideal gas with adiabatic exponent γ undergoes a process in which its pressure P is related to its volume V by the relation $P = P_0 - \alpha V$, where P_0 and α are positive constants. The volume starts from being very close to zero and increases monotonically to $\frac{P_0}{\alpha}$. At what value of the volume during the process does the gas have maximum entropy?

- (a) $\frac{P_0}{\alpha(1+\gamma)}$ (b) $\frac{\gamma P_0}{\alpha(1-\gamma)}$ (c) $\frac{\gamma P_0}{\alpha(1+\gamma)}$ (d) $\frac{P_0}{\alpha(1-\gamma)}$

Ans.: (c)

Solution: $TdS = nC_v dT + PdV \Rightarrow TdS = \frac{nRdT}{(\gamma-1)} + PdV$

For maximum entropy, $dS = 0$

For Ideal gas, $PV = nRT \Rightarrow PdV + VdP = nRdT$

$$\Rightarrow TdS = \frac{PdV + VdP}{(\gamma-1)} + PdV \Rightarrow \frac{PV}{nR} dS = \frac{\gamma}{(\gamma-1)} PdV + \frac{VdP}{(\gamma-1)}$$

Since, $P = P_0 - \alpha V \Rightarrow dP = -\alpha dV$

$$\frac{PV}{nR} dS = \frac{\gamma}{(\gamma-1)} PdV - \frac{\alpha VdV}{(\gamma-1)} \Rightarrow \frac{dS}{dV} = \frac{\gamma nRP}{(\gamma-1)PV} - \frac{nR}{(\gamma-1)PV} \alpha V$$

For maximum entropy, $\frac{dS}{dV} = 0 \Rightarrow \gamma P - \alpha V = 0 \Rightarrow \gamma(P_0 - \alpha V) = \alpha V$

$$\Rightarrow V = \frac{\gamma P_0}{\alpha(1+\gamma)}$$

Q26. An ideal gas has a specific heat ratio $\frac{C_p}{C_v} = 2$. Starting at a temperature T_1 the gas under

goes an isothermal compression to increase its density by a factor of two. After this an adiabatic compression increases its pressure by a factor of two. The temperature of the gas at the end of the second process would be:

- (a) $\frac{T_1}{2}$ (b) $\sqrt{2}T_1$ (c) $2T_1$ (d) $\frac{T_1}{\sqrt{2}}$

Ans.: (b)

Solution: During the isothermal process, $T = T_1$ is constant

Let us assume, the adiabatic process started at point $A (P_1, T_1)$ and at point B the

coordinate is (P_2, T_2) , it is given $P_1^{1-\gamma} T_1^\gamma = P_2^{1-\gamma} T_2^\gamma \Rightarrow T_2 = \left(\frac{P_1}{P_2}\right)^{\frac{1-\gamma}{\gamma}} T_1 \Rightarrow T_2 = \left(\frac{P_1}{2P_1}\right)^{\frac{1-2}{2}} T_1$

$$\Rightarrow T_2 = \sqrt{2} T_1$$

Q27. A two dimensional box in a uniform magnetic field B contains $\frac{N}{2}$ localised spin- $\frac{1}{2}$ particles with magnetic moment μ , and $\frac{N}{2}$ free spinless particles which do not interact with each other. The average energy of the system at a temperature T is:

- (a) $3NkT - \frac{1}{2} N \mu B \sinh\left(\frac{\mu B}{k_B T}\right)$ (b) $NkT - \frac{1}{2} N \mu B \tanh\left(\frac{\mu B}{k_B T}\right)$
 (c) $\frac{1}{2} NkT - \frac{1}{2} N \mu B \tanh\left(\frac{\mu B}{k_B T}\right)$ (d) $\frac{3}{2} NkT + \frac{1}{2} N \mu B \cosh\left(\frac{\mu B}{k_B T}\right)$

Ans.: (c)

Solution: For $\frac{N}{2}$ free particles in two dimension, average energy is $\frac{N}{2} kT$ and for $\frac{N}{2}$ localized

spin- $\frac{1}{2}$ particle, the average energy is $-\frac{1}{2} N \mu B \tanh\left(\frac{\mu B}{k_B T}\right)$

Then average energy of system at temperature T is

$$\langle E \rangle = \frac{NkT}{2} - \frac{1}{2} N \mu B \tanh\left(\frac{\mu B}{k_B T}\right).$$

Q28. A gas of N molecules of mass m is confined in a cube of volume $V = L^3$ at temperature T . The box is in a uniform gravitational field $-g\hat{z}$. Assume that the potential energy of a molecule is $U = mgz$ where $z \in [0, L]$ is the vertical coordinate inside the box. The pressure $P(z)$ at height z is:

$$(a) P(z) = \frac{N mgL}{V} \frac{\exp\left(-\frac{mg\left(z - \frac{L}{2}\right)}{k_B T}\right)}{\sinh\left(\frac{mgL}{2k_B T}\right)}$$

$$(b) P(z) = \frac{N mgL}{V} \frac{\exp\left(-\frac{mg\left(z - \frac{L}{2}\right)}{k_B T}\right)}{\cosh\left(\frac{mgL}{2k_B T}\right)}$$

$$(c) P(z) = \frac{k_B T N}{V}$$

$$(d) P(z) = \frac{N}{V} mgz$$

Ans.: (c)

Solution: The partition function of a system is given by,

$$Z_N = \left(\frac{2\pi mk_B T}{h^2}\right)^{\frac{3N}{2}} \left(\frac{k_B T V}{mgL}\right)^N \left(1 - \exp\left(-\frac{mgL}{k_B T}\right)\right)^N$$

Helmholtz free energy is given by, $F = -k_B T \ln Z_N$

$$\text{Pressure is given by } P = -\left(\frac{\partial F}{\partial V}\right)_{T, N} = \frac{k_B T N}{V}$$

JEST 2017

Part-A: 1-Mark Questions

- Q29. After the detonation of an atom bomb, the spherical ball of gas was found to be of 15 meter radius at a temperature of $3 \times 10^5 K$. Given the adiabatic expansion coefficient $\gamma = 5/3$, what will be the radius of the ball when its temperature reduces to $3 \times 10^3 K$?
- (a) 156m (b) 50m (c) 150m (d) 100m

Ans. : (c)

$$\text{Solution: } T_1 V_1^{\gamma-1} = T_2 V_2^{\gamma-1} \Rightarrow V_2 = \left(\frac{T_1}{T_2}\right)^{\frac{1}{\gamma-1}} V_1 \Rightarrow V_2 = \left(\frac{T_1}{T_2}\right)^{3/2} V_1$$

$$\Rightarrow R = \left(\frac{T_1}{T_2}\right)^{\frac{1}{2}} V_1 \Rightarrow R = \left(\frac{3 \times 10^5}{3 \times 10^3}\right)^{\frac{1}{2}} 15 = 150$$

- Q30. If the mean square fluctuations in energy of a system in equilibrium at temperature T is proportional to T^α , then the energy of the system is proportional to
- (a) $T^{\alpha-2}$ (b) $T^{\frac{\alpha}{2}}$ (c) $T^{\alpha-1}$ (d) T^α

Ans. : (c)

$$\text{Solution: } (\Delta E)^2 = kT^2 C_V \Rightarrow T^{\alpha-2} \propto C_V \Rightarrow T^{\alpha-2} \propto \left(\frac{\partial U}{\partial T}\right)_V \Rightarrow U \propto T^{\alpha-1}$$

- Q31. Suppose that the number of microstates available to a system of N particles depends on N and the combined variable UV^2 , where U is the internal energy and V is the volume of the system. The system initially has volume $2m^3$ and energy $200 J$. It undergoes an isentropic expansion to volume $4m^3$. What is the final pressure of the system in SI units?

Ans. : 25

$$\text{Solution: Here, } \Omega = (UV^2)^N \Rightarrow S = Nk \ln(UV^2)$$

From law of thermodynamics,

$$TdS = dU + PdV$$

$$\Rightarrow \left.\frac{\partial S}{\partial U}\right|_V = \frac{1}{T} \Rightarrow U = NkT \quad \dots\dots(1)$$

and $\left. \frac{\partial S}{\partial V} \right|_U = \frac{P}{T} \Rightarrow PV = 2NkT$ (2)

From equation (1) and (2),

$$PV = 2U \quad \text{.....(3)}$$

Now, from equation (3),

$$P_i V_i = 2U_i \Rightarrow P_i = \frac{2U_i}{V_i} = \frac{2 \times 200}{2} = 200 \text{ atm} \quad \text{.....(4)}$$

As the given process is isoentropic,

$$\therefore TdS = 0 = dU + PdV \Rightarrow \left. \frac{\partial U}{\partial V} \right|_S = -P$$

and from equation (3),

$$\left. \frac{\partial U}{\partial V} \right|_S = \frac{P}{2} + \frac{V}{2} \left. \frac{\partial P}{\partial V} \right|_S = -P \Rightarrow \frac{V}{2} \frac{\partial P}{\partial V} = -\frac{3P}{2} \Rightarrow \frac{\partial P}{P} = -3 \frac{\partial V}{V}$$

On solving above equation, we have

$$PV^3 = K \quad (\text{constant})$$

$$\Rightarrow P_f V_f^3 = P_i V_i^3 \Rightarrow P_f = \frac{P_i V_i^3}{V_f^3} = \frac{200 \times 2^3}{4^3} = 25 \text{ atm}$$

Q32. A cylinder at temperature $T = 0$ is separated into two compartments A and B by a free sliding piston. Compartments A and B are filled by Fermi gases made of spin $1/2$ and $3/2$ particles respectively. If particles in both the compartments have same mass, the ratio of equilibrium density of the gas in compartment A to that of gas in compartment B is

- (a) 1 (b) $\frac{1}{3^{2/5}}$ (c) $\frac{1}{2^{2/5}}$ (d) $\frac{1}{2^{2/3}}$

Ans. : (c)

Solution: **Follow Pathria** Page 198 equation 20 for ϵ_f

And equation (38) at pages 200

From equation (38) at $T = 0$

$$p = \frac{2}{5} n \epsilon_f$$

$$= \frac{2}{5} n \left(\frac{6\pi^2 n}{g} \right)^{2/3} \frac{\hbar^2}{2m} \quad (\text{using equation (24)})$$

for equilibrium $\rho_A = \rho_B$

$$\Rightarrow n_A \left(\frac{n_A}{g_A} \right)^{2/3} = n_B \left(\frac{n_B}{g_B} \right)^{2/3}$$

$$\left(\frac{n_A}{n_B} \right)^{5/3} = \left(\frac{g_A}{g_B} \right)^{2/3}$$

$$g_A = 25 + 1 = 2 \times \frac{1}{2} + 1 = 2, \quad g_B = 25 + 1 = 2 \times \frac{5}{2} + 1 = 4$$

$$\left(\frac{g_A}{g_B} \right) = \left(\frac{1}{2} \right)^{2/3} \Rightarrow \frac{n_A}{n_B} = \left(\frac{1}{2} \right)^{2/3 \times 2/3}$$

$$\frac{n_A}{n_B} = \frac{1}{2^{2/5}}$$

