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Forum for CSIR-UGC JRF/NET, GATE, IIT-JAM/IISc, JEST, TIFR and GRE in PHYSICAL SCIENCES

Basic Mathematics Formula Sheet for Physical Sciences

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1. Trigonometry

1.1 Trigonometrical Ratios and Identities

1. $\sin^2\theta+\cos^2\theta=1$
2. $\sec^2\theta=1+\tan^2\theta$
3. $\operatorname{cosec}^2\theta=1+\cot^2\theta$
4. $\tan\theta=\dfrac{\sin\theta}{\cos\theta}$
5. $\cot\theta=\dfrac{\cos\theta}{\sin\theta}$
6. $\sin\theta=\dfrac{1}{\operatorname{cosec}\theta}$
7. $\cos\theta=\dfrac{1}{\sec\theta}$
8. $\tan\theta=\dfrac{1}{\cot\theta}$

Addition and Subtraction Formulae

For any two angles A and B

1. $Sin(A+B)=\sin A\cos B+\cos A\sin B$
2. $Sin(A-B)=\sin A\cos B-\cos A\sin B$
3. $\cos(A+B)=\cos A\cos B-\sin A\sin B$
4. $\cos(A-B)=\cos A\cos B+\sin A\sin B$
5. $\tan(A+B)=\dfrac{\tan A+\tan B}{1-\tan A.\tan B}$
6. $\tan(A-B)=\dfrac{\tan A-\tan B}{1+\tan A.\tan B}$

Double Angle Formulae

1. $\sin 2\theta=2\sin\theta\cos\theta$,
2. $\cos 2\theta=\cos^2\theta-\sin^2\theta=1-2\sin^2\theta=2\cos^2\theta-1$
3. $\tan 2\theta=\dfrac{2\tan\theta}{1-\tan^2\theta}$

Triple angle Formulae

1. $\sin 3\theta=3\sin\theta-4\sin^3\theta$
2. $\cos 3\theta=4\cos^3\theta-3\cos\theta$
3. $\tan 3\theta=\dfrac{3\tan\theta-\tan^3\theta}{1-3\tan^2\theta}$

Trigonometric Ratios of $\theta/2$

1. $\sin\theta=2\sin\dfrac{\theta}{2}\cos\dfrac{\theta}{2}$,
2. $\cos\theta=\cos^2\dfrac{\theta}{2}-\sin^2\dfrac{\theta}{2}=2\cos^2\dfrac{\theta}{2}=1-2\sin^2\dfrac{\theta}{2}$
3. $\tan\theta=\dfrac{2\tan\frac{\theta}{2}}{1-\tan^2\frac{\theta}{2}}$

Formulae for sin2θ & cos2θ in terms of tanθ

1. $\sin 2\theta = \dfrac{2\tan\theta}{1+\tan^2\theta}$

2. $\cos 2\theta = \dfrac{1-\tan^2\theta}{1+\tan^2\theta}$

Formulae for sinθ & cosθ in terms of tanθ/2

1. $\sin\theta = \dfrac{2\tan\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}}$

2. $\cos\theta = \dfrac{1-\tan^2\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}}$

Transformation of sum/differences into Products

1. $\sin C + \sin D = 2\sin\left(\dfrac{C+D}{2}\right)\cos\left(\dfrac{C-D}{2}\right)$

2. $\sin C - \sin D = 2\cos\left(\dfrac{C+D}{2}\right)\sin\left(\dfrac{C-D}{2}\right)$

3. $\cos C + \cos D = 2\cos\left(\dfrac{C+D}{2}\right)\cos\left(\dfrac{C-D}{2}\right)$

4. $\cos C - \cos D = -2\sin\left(\dfrac{C+D}{2}\right)\sin\left(\dfrac{C-D}{2}\right) = 2\sin\left(\dfrac{C+D}{2}\right)\sin\left(\dfrac{D-C}{2}\right)$

Transformations of Products into sum/difference

1. $2 Sin A\cos B = Sin(A+B) + Sin(A-B)$

2. $2\cos A\sin B = Sin(A+B) - Sin(A-B)$

3. $2\cos A\cos B = \cos(A+B) + \cos(A-B)$

4. $2\sin A\sin B = \cos(A-B) - \cos(A+B)$

Trigonometric Ratios of (-θ)

1. $\sin(-\theta) = -\sin\theta$

2. $\cos(-\theta) = \cos\theta$

3. $\tan(-\theta) = -\tan\theta$

4. $\cot(-\theta) = -\theta$

5. $\sec(-\theta) = -\cot\theta$

6. $\cos ec(-\theta) = -\cos ec\theta$

Trigonometric Ratio of $\left(\frac{\pi}{2}-\theta\right)$ **:** (All Positive)

1. $\cos\left(\frac{\pi}{2}-\theta\right)=\sin\theta$
2. $\sin\left(\frac{\pi}{2}-\theta\right)=\cos\theta$
3. $\tan\left(\frac{\pi}{2}-\theta\right)=\cot\theta$
4. $\cot\left(\frac{\pi}{2}-\theta\right)=\tan\theta$
5. $\operatorname{cosec}\left(\frac{\pi}{2}-\theta\right)=\sec\theta$
6. $\sec\left(\frac{\pi}{2}-\theta\right)=\operatorname{cosec}\theta$

Trigonometric Ratio of $\left(\frac{\pi}{2}+\theta\right)$**:**(Only $\sin\theta$ and $\operatorname{cosec}\theta$ is Positive)

1. $\cos\left(\frac{\pi}{2}+\theta\right)=-\sin\theta$
2. $\sin\left(\frac{\pi}{2}+\theta\right)=\cos\theta$
3. $\tan\left(\frac{\pi}{2}+\theta\right)=-\cot\theta$
4. $\cot\left(\frac{\pi}{2}+\theta\right)=-\tan\theta$
5. $\operatorname{cosec}\left(\frac{\pi}{2}+\theta\right)=\sec\theta$
6. $\sec\left(\frac{\pi}{2}+\theta\right)=-\operatorname{cosec}\theta$

Trigonometric Ratios of $(\pi-\theta)$**:**(Only $\sin\theta$ and $\operatorname{cosec}\theta$ is Positive)

1. $\cos(\pi-\theta)=-\cos\theta$
2. $\sin(\pi-\theta)=\sin\theta$
3. $\tan(\pi-\theta)=-\tan\theta$
4. $\cot(\pi-\theta)=-\cot\theta$
5. $\operatorname{cosec}(\pi-\theta)=\operatorname{cosec}\theta$
6. $\sec(\pi-\theta)=-\sec\theta$

Trigonometric Ratios of $(\pi+\theta)$**:**(Only $\tan\theta$ and $\cot\theta$ is Positive)

1. $\cos(\pi+\theta)=-\cos\theta$
2. $\sin(\pi+\theta)=-\sin\theta$
3. $\tan(\pi+\theta)=\tan\theta$
4. $\cot(\pi+\theta)=\cot\theta$
5. $\operatorname{cosec}(\pi+\theta)=-\operatorname{cosec}\theta$
6. $\sec(\pi+\theta)=-\sec\theta$

Trigonometric Ratio of $\left(\frac{3\pi}{2}-\theta\right)$**:**(Only $\tan\theta$ and $\cot\theta$ is Positive)

1. $\cos\left(\frac{3\pi}{2}-\theta\right)=-\sin\theta$
2. $\sin\left(\frac{3\pi}{2}-\theta\right)=-\cos\theta$
3. $\tan\left(\frac{3\pi}{2}-\theta\right)=\cot\theta$
4. $\cot\left(\frac{3\pi}{2}-\theta\right)=\tan\theta$
5. $\operatorname{cosec}\left(\frac{3\pi}{2}-\theta\right)=-\sec\theta$
6. $\sec\left(\frac{3\pi}{2}-\theta\right)=-\operatorname{cosec}\theta$

Trigonometric Ratio of $\left(\frac{3\pi}{2}+\theta\right)$**:**(Only $\cos\theta$ and $\sec\theta$ is Positive)

1. $\cos\left(\frac{3\pi}{2}+\theta\right)=\sin\theta$
2. $\sin\left(\frac{3\pi}{2}+\theta\right)=-\cos\theta$
3. $\tan\left(\frac{3\pi}{2}+\theta\right)=-\cot\theta$
4. $\cot\left(\frac{3\pi}{2}+\theta\right)=-\tan\theta$
5. $\operatorname{cosec}\left(\frac{3\pi}{2}+\theta\right)=-\sec\theta$
6. $\sec\left(\frac{3\pi}{2}+\theta\right)=\operatorname{cosec}\theta$

Trigonometric Ratios of $(2\pi-\theta)$**:**(Only $\cos\theta$ and $\sec\theta$ is Positive)

1. $\cos(2\pi-\theta)=\cos\theta$
2. $\sin(2\pi-\theta)=-\sin\theta$
3. $\tan(2\pi-\theta)=-\tan\theta$
4. $\cot(2\pi-\theta)=-\cot\theta$
5. $\operatorname{cosec}(2\pi-\theta)=-\operatorname{cosec}\theta$
6. $\sec(2\pi-\theta)=\sec\theta$

Trigonometric Ratios of $(2\pi+\theta)$**:** (All Positive)

1. $\cos(2\pi+\theta)=\cos\theta$
2. $\sin(2\pi+\theta)=\sin\theta$
3. $\tan(2\pi+\theta)=\tan\theta$
4. $\cot(2\pi+\theta)=\cot\theta$
5. $\operatorname{cosec}(2\pi+\theta)=\operatorname{cosec}\theta$
6. $\sec(2\pi+\theta)=\sec\theta$

Short-cut method to remember the Trigonometric ratios

1. $\sin\left(\frac{n\pi}{2}\pm\theta\right)=\pm\sin\theta$

2. $\cos\left(\frac{n\pi}{2}\pm\theta\right)=\pm\cos\theta$ when n is an even integer

3. $\tan\left(\frac{n\pi}{2}\pm\theta\right)=\pm\tan\theta$

4. $\sin\left(\frac{n\pi}{2}\pm\theta\right)=\pm\cos\theta$

5. $\cos\left(\frac{n\pi}{2}\pm\theta\right)=\pm\sin\theta$ when n is an odd integer

6. $\tan\left(\frac{n\pi}{2}\pm\theta\right)=\pm\cot\theta$

θ	0^o	30^o	45^o	60^o	90^o	120^o	135^o	150^o	180^o	270^o	360^o
$\sin\theta$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	-1	0
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{1}{\sqrt{2}}$	$-\frac{\sqrt{3}}{2}$	-1	0	1
$\tan\theta$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	∞	$-\sqrt{3}$	-1	$-\frac{1}{\sqrt{3}}$	0	∞	0
$\cot\theta$	∞	$\sqrt{3}$	1	$\frac{1}{\sqrt{3}}$	0	$\frac{-1}{\sqrt{3}}$	-1	$-\sqrt{3}$	∞	0	∞
$\sec\theta$	1	$\frac{2}{\sqrt{3}}$	$\sqrt{2}$	2	∞	-2	$-\sqrt{2}$	$\frac{-2}{\sqrt{3}}$	-1	∞	1
$\text{cosec}\,\theta$	∞	2	$\sqrt{2}$	$\frac{2}{\sqrt{3}}$	1	$\frac{2}{\sqrt{3}}$	$\sqrt{2}$	2	∞	-1	∞

1.2 Inverse Circular Functions

1. $\sin^{-1}(\sin x) = x$
2. $\cos^{-1}(\cos x) = x$
3. $\tan^{-1}(\tan x) = x$
4. $\cot^{-1}(\cot x) = x$
5. $\sec^{-1}(\sec x) = x$
6. $\operatorname{cosec}^{-1}(\operatorname{cosec}) = x$
7. $\sin(\sin^{-1} x) = x$
8. $\cos(\cos^{-1} x) = x$
9. $\sec(\sec^{-1} x) = x$
10. $\operatorname{cosec}(\operatorname{cosec}^{-1} x) = x$
11. $\sin^{-1}\left(\frac{1}{x}\right) = \operatorname{cosec}^{-1} x$
12. $\cos^{-1}\left(\frac{1}{x}\right) = \sec^{-1} x$
13. $\tan^{-1}\left(\frac{1}{x}\right) = \cot^{-1} x$
14. $\cot^{-1}\left(\frac{1}{x}\right) = \tan^{-1} x$
15. $\sec^{-1}\left(\frac{1}{x}\right) = \cos^{-1} x$
16. $\operatorname{cosec}^{-1}\left(\frac{1}{x}\right) = \sin^{-1} x$
17. $\sin^{-1}(-x) = -\sin^{-1} x$
18. $\cos^{-1}(-x) = \pi - \cos^{-1} x$
19. $\tan^{-1}(-x) = -\tan^{-1} x$
19. $\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$
20. $\tan^{-1} x + \cot^{-1} x = \frac{\pi}{2}$
21. $\sec^{-1} x + \operatorname{cosec}^{-1} x = \frac{\pi}{2}$
22. $\sin^{-1}\sqrt{1-x^2} = \cos^{-1} x$
23. $\cos^{-1}\sqrt{1-x^2} = \sin^{-1} x$
24. $\tan^{-1}\sqrt{x^2-1} = \sec^{-1} x$
25. $\cot^{-1}\sqrt{x^2-1} = \operatorname{cosec}^{-1} x$
26. $\sec^{-1}\sqrt{1+x^2} = \tan^{-1} x$
27. $\operatorname{cosec}^{-1}\sqrt{1+x^2} = \cot^{-1} x$
28. $\sin^{-1}\left(2x\sqrt{1-x^2}\right) = 2\sin^{-1} x$
29. $\sin^{-1}(3x-4x^3) = 3\sin^{-1} x$
30. $\cos^{-1}(4x^3-3x) = 3\cos^{-1} x$
31. $\tan^{-1}\left(\frac{2x}{1-x^2}\right) = 2\tan^{-1} x$
32. $\tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right) = 3\tan^{-1} x$
33. $\tan^{-1}\left(\frac{x+y}{1-xy}\right) = \tan^{-1} x + \tan^{-1} y$
34. $\tan^{-1}\left(\frac{x-y}{1+xy}\right) = \tan^{-1} x - \tan^{-1} y$

Some Important Expansions: 1. $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - ....$

2. $\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + ...$

3. $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - ....$

4. $\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + ...$

5. $\tan x = x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + ...$

6. $e^x = 1 + x + \frac{x^2}{2!} + \frac{x3}{3!} + ...$

7. $e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x3}{3!} + ...$

Some useful substitutions:-

Expressions	Substitution	Formula	Result
$3x - 4x^3$	$x = \sin\theta$	$3\sin\theta - 4\sin^3\theta$	Sin3θ
$4x^3 - 3x$	$x = \cos\theta$	$4\cos^3\theta - 3\cos\theta$	cos3θ
$\frac{3x - x^3}{1 - 3x^2}$	$x = \tan\theta$	$\frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}$	tan3θ
$\frac{2x}{1 + x^2}$	$x = \tan\theta$	$\frac{2\tan\theta}{1 + \tan^2\theta}$	sin2θ
$\frac{1 - x^2}{1 + x^2}$	$x = \tan\theta$	$\frac{1 - \tan^2\theta}{1 + \tan^2\theta}$	cos2θ
$\frac{2x}{1 - x^2}$	$x = \tan\theta$	$\frac{2\tan\theta}{1 - \tan^2\theta}$	tan2θ
$1 - 2x^2$	$x = \sin\theta$	$1 - 2\sin^2\theta$	cos2θ
$2x^2 - 1$	$x = \cos\theta$	$2\cos^2\theta - 1$	cos2θ
$1 - x^2$	$x = \sin\theta$	$1 - \sin^2\theta$	$\cos^2\theta$
$1 - x^2$	$x = \cos\theta$	$1 - \cos^2\theta$	$\sin^2\theta$
$x^2 - 1$	$x = \sec\theta$	$\sec^2\theta - 1$	$\tan^2\theta$
$x^2 - 1$	$x = \mathrm{cosec}\,\theta$	$\cos ec^2\theta - 1$	$\cot^2\theta$
$1 + x^2$	$x = \tan\theta$	$1 + \tan^2\theta$	$\sec^2\theta$
$1 + x^2$	$x = \cot\theta$	$1 + \cot^2\theta$	$\mathrm{cosec}^2\theta$

2. Differential and integral Calculus

2.1 Differentiation

1. $f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$

2. $f'(a)=\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$

3. $\frac{dy}{dx}=\lim_{\delta x\to 0}\frac{\delta y}{\delta x}$

4. $\frac{d}{dx}(k)=0$; k is constant function

5. $\frac{d}{dx}(x)=1$

6. $\frac{d}{dx}(\sqrt{x})=\frac{1}{2\sqrt{x}}$

7. $\frac{d}{dx}\left(\frac{1}{x^n}\right)=\frac{-n}{x^{n+1}}$

8. $\frac{d}{dx}(x^n)=nx^{n-1};n\in N$

9. $\frac{d}{dx}\left(\frac{1}{x}\right)=\frac{-1}{x^2}$

10. $\frac{d}{dx}(\sin x)=\cos x$

11. $\frac{d}{dx}(\cos x)=-\sin x$

12. $\frac{d}{dx}(\tan x)=\sec^2 x$

13. $\frac{d}{dx}(\sec x)=\sec x.\tan x$

14. $\frac{d}{dx}(\cos ecx)=-\cos ecx.\cot x$

15. $\frac{d}{dx}(a^x)=a^x\log a;(a>0,a\neq 1)$

16. $\frac{d}{dx}(e^x)=e^x$

17. $\frac{d}{dx}(\log x)=\frac{1}{x}$

18. $\frac{d}{dy}(\sin^{-1}x)=\frac{1}{\sqrt{1-x^2}}$; $-1\le x\le 1$

19. $\frac{d}{dx}(\cos^{-1}x)=\frac{-1}{\sqrt{1-x^2}};-1\le x\le 1$

20. $\frac{d}{dx}(\tan^{-1}x)=\frac{1}{1+x^2};\quad x\in R$

21. $\frac{d}{dx}(\cot^{-1}x)=\frac{-1}{1+x^2};\quad x\in R$

22. $\frac{d}{dx}(\sec^{-1})=\frac{1}{x\sqrt{x^2-1}}:\qquad |x|\ge 1$

23. $\frac{d}{dx}(\cos ec^{-1}x)=\frac{-1}{x\sqrt{x^2-1}};\qquad |x|\ge 1$

Rules of Differentiations

1. Addition Rule: If $y = (u + v)$ then $\Rightarrow \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$

2. Substations Rule: If $y = (u - v)$ then $\Rightarrow \frac{dy}{dx} = \frac{du}{dx} - \frac{dv}{dx}$

3. Product Rule: If $y = uv$ then $\Rightarrow \frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$

4. Quotient Rule: If $y = \frac{u}{v}$ then $\Rightarrow \frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

5. If $y = f(u)$ is $u = g(x)$ then $\frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dx}$

6. If $u = f(y)$, then $\frac{du}{dx} = \frac{du}{dy}.\frac{dy}{dx} = f'(y)\frac{dy}{dx}$

7. $\frac{dy}{dx}.\frac{dx}{dy} = 1$ or $\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$ where $\frac{dx}{dy} \neq 0$

Derivatives of composite functions

1. $\frac{d}{dx}[f(x)]^n = n[f(x)]^{n-1}.\frac{d}{dx}[f(x)]$

2. $\frac{d}{dx}\left[\sqrt{f(x)}\right] = \frac{1}{2\sqrt{f(x)}}\frac{d}{dx}[f(x)]$

3. $\frac{d}{dx}\left[\frac{1}{f(x)}\right] = \frac{-1}{[f(x)]^2}.\frac{d}{dx}[f(x)]$

4. $\frac{d}{dx}[\sin f(x)] = \cos f(x).\frac{d}{dx}[f(x)]$

5. $\frac{d}{dx}[\cos f(x)] = -\sin f(x).\frac{d}{dx}[f(x)]$

6. $\frac{d}{dx}[\tan f(x)] = \sec^2 f(x).\frac{d}{dx}[f(x)]$

7. $\frac{d}{dx}[\cot f(x)] = -\cos ec^2 f(x).\frac{d}{dx}[f(x)]$

8. $\frac{d}{dx}[\sec f(x)] = \sec f(x)\tan f(x).\frac{d}{dx}[f(x)]$

9. $\frac{d}{dx}[\cos ecf(x)] = -\cos ecf(x)\cot f(x).\frac{d}{dx}[f(x)]$

10. $\frac{d}{dx}[\log f(x)] = \frac{1}{f(x)}.\frac{d}{dx}[f(x)]$

11. $\frac{d}{dx}\left[a^{f(x)}\right] = a^{f(x)}\log a.\frac{d}{dx}[f(x)]$

12. $\frac{d}{dx}\left[e^{f(x)}\right]=e^{f(x)}\frac{d}{dx}[f(x)]$

13. $\frac{d}{dx}[f(g(x))]^n=n[f(g(x))]^{n-1}[f'(g(x))]\frac{d}{dx}(g(x))$

Derivatives of composite functions

1. $\frac{d}{dx}\left[\sin^{-1} f(x)\right]=\frac{1}{\sqrt{1-[f(x)]^2}}\cdot\frac{d}{dx}[f(x)]$

2. $\frac{d}{dx}\left[\cos^{-1} f(x)\right]=\frac{-1}{\sqrt{1-[f(x)]^2}}\cdot\frac{d}{dx}[f(x)]$

3. $\frac{d}{dx}\left[\tan^{-1} f(x)\right]=\frac{1}{1+[f(x)]^2}\cdot\frac{d}{dx}[f(x)]$

4. $\frac{d}{dx}\left[\cot^{-1} f(x)\right]=\frac{-1}{1+[f(x)]^2}\cdot\frac{d}{dx}[f(x)]$

5. $\frac{d}{dx}\left[\sec^{-1} f(x)\right]=\frac{1}{f(x)\sqrt{[f(x)]^2-1}}\cdot\frac{d}{dx}[f(x)]$

6. $\frac{d}{dx}\left[\operatorname{cosec}^{-1} f(x)\right]=\frac{-1}{f(x)\sqrt{[f(x)]^2-1}}\cdot\frac{d}{dx}[f(x)]$

Implicit functions:-

Take the derivatives of these functions directly and find dy/dx

Parametric functions:-

If $x=f(t)\,\&\,y=g(t)$ then $\frac{dy}{dx}=\frac{dy/dx}{dx/dt}$ where $\frac{dx}{dt}\neq 0$

Logarithemic Differentiation:- If the function is in the form of $[f(x)]^{g(x)}$

Then taking Logarithm on both sides 1^5 & then find dy/dx

Higher order Derivatives of composite functions:-

$y_2=\frac{d^2y}{dx^2}=f''(x)$ IInd order, $y_3=\frac{d^3y}{dx^3}=f'''(x)$ IIIrd order

In General; $y_n=\frac{d^ny}{dx^n}=f^n(x)$ nth order

2.2 Limits

Limits of function

If for every $\varepsilon > 0$ there exist $\delta > 0$ such that if $|f(x) - l| < \varepsilon$ whenever $0 < |x - a| < \delta$ then we say lim of $f(x)$ as $x \to a$ is l

i.e. $\lim_{x\to a} f(x) = l$

Theorem of limits

If $f(x)$ and $g(x)$ are two functions then

1. $\lim_{x\to a}[f(x) + g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$
2. $\lim_{x\to a}[f(x) - g(x)] = \lim_{x\to a} f(x) - \lim_{x\to a} g(x)$
3. $\lim_{x\to a}[f(x).g(x)] = \left(\lim_{x\to a} f(x)\right)\left(\lim_{x\to a} g(x)\right)$
4. $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$
5. $\lim_{x\to a}[kf(x)] = k\left(\lim_{x\to a} f(x)\right)$ where k is constant
6. $\lim_{x\to a} \sqrt{f(x)} = \sqrt{\lim_{x\to a} f(x)}$
7. $\lim_{x\to a}[f(x)]^{p/q} = \left[\lim_{x\to a} f(x)\right]^{p/q}$: where p & q are integers

Some Important standard limits

1. $\lim_{x\to a} x = a$
2. $\lim_{x\to a} c = c$: where c is constant $c \in R$
3. $\lim_{x\to a} x^n = a^n; \quad n \in R$
4. $\lim_{x\to a} \frac{x^n - a^n}{x - a} = na^{n-1}; \quad n \in N, a > 0$
5. $\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1$
6. $\lim_{\theta\to 0} \frac{\theta}{\sin\theta} = 1$
7. $\lim_{\theta\to 0} \frac{\tan\theta}{\theta} = 1$
8. $\lim_{\theta\to 0} \frac{\theta}{\tan\theta} = 1$
9. $\lim_{\theta\to 0} \frac{\sin k\theta}{\theta} = k$
10. $\lim_{\theta\to 0} \frac{\tan k\theta}{\theta} = k$

11. $\lim\limits_{x\to 0}\frac{1-\cos x}{x}=\frac{1}{2}$

12. $\lim\limits_{x\to 0}\frac{x^2}{1-\cos x}=2$

13. $\lim\limits_{x\to 0}\frac{\sin x^0}{x}=\frac{\pi}{18};\lim\limits_{x\to 0}\frac{\sin mx^0}{\sin nx^0}=\frac{m}{n}$

14. $\lim\limits_{x\to 0}\frac{1-\cos kx}{x^2}=\frac{k^2}{2}$

15. $\lim\limits_{x\to 0}\cos x=1;\quad \lim\limits_{x\to \pi/2}\cos x=0$

16. $\lim\limits_{x\to 0}\sin x=0;\quad \lim\limits_{x\to \pi/2}\sin \pi/2=1$

17. $\lim\limits_{x\to a}\frac{a^x-1}{x}=\log a$ where $a>0$

18. $\lim\limits_{x\to a}\frac{e^x-1}{x}=1$

19. $\lim\limits_{x\to a}\frac{\log(1+x)}{x}=1;\quad \lim\limits_{x\to a}\frac{\log_a(1+x)}{x}=\log a^e\quad a>0$

20. $\lim\limits_{x\to 0}(1+x)^{1/x}=e;\quad \lim\limits_{x\to 0}(1+kx)^{1/x}=e^k$

21. $\lim\limits_{x\to 0}\frac{\log(1+kx)}{x}=k$

22. $\lim\limits_{x\to 0}\frac{\cos ax-\cos bx}{\cos cx-\cos dx}=\frac{a^2-b^2}{c^2-d^2}$

23. $\lim\limits_{x\to 0}\left(\frac{\cos ax-\cos bx}{x}\right)=\frac{b^2-a^2}{2}$

24. $\lim\limits_{x\to 0}\left(\frac{1+ax}{1+bx}\right)^{1/x}=e^{a-b}$

25. $\lim\limits_{x\to 0}\left(\frac{a+bx}{a+cx}\right)^{1/x}=e^{\frac{b-c}{a}}$

26. $\lim\limits_{x\to \infty}\frac{1}{x}=0;\ \lim\limits_{x\to -\infty}\frac{1}{x}=0$

27. $\lim\limits_{x\to \infty}\frac{1}{x^2}=0;\ \lim\limits_{x\to -\infty}\frac{1}{x^2}=0$

28. $\lim\limits_{x\to \infty}\frac{1}{x^k}=0$ where $k>0$

29. $\lim\limits_{x\to \infty}k=k;\quad \lim\limits_{x\to -\infty}k=k$ where k is constant

30. $\lim\limits_{x\to a}\sin x=\sin a$

31. $\lim\limits_{x\to a}\cos x=\cos a$

32. $\lim\limits_{x\to 1}\frac{x+x^2+x^3+\ldots x^n-n}{x-1}=\frac{n(n+1)}{2}$

33. $\lim\limits_{x\to 0}\frac{a^x-b^x}{x}=\log\left(\frac{a}{b}\right);\quad a,b>0$

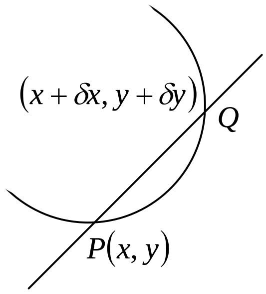
34. $\lim\limits_{x\to 0}\frac{\sec x-1}{x^2}=\frac{1}{2};\lim\limits_{x\to 0}\frac{\cos ecx-1}{x^2}=1$

35. $\lim\limits_{x\to \infty}\left(1+\frac{1}{x^2}\right)^x=e^x;\lim\limits_{n\to \infty}\left(1+\frac{1}{n}\right)^n=e$

36. $\lim\limits_{h\to \infty}\left(1+\frac{a}{h}\right)^h=e^a$

2.3 Tangents and Normal

Tangent at (x, y) to $y = f(x)$



Let $y = f(x)$ be a given curve and $P(x, y)$ and $Q(x+\delta x, y+\delta y)$ be two neighbouring points on it.

Equation of the line PQ is

$$Y - y = \frac{y+\delta y - y}{x+\delta x - x}(X - x) \text{ or } Y - y = \frac{\delta y}{\delta x}(X - x)$$

This line will be tangent to the given curve at P if $Q \to P$ which in tern means that $\delta x \to 0$ and we know that

$$\lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \frac{dy}{dx}$$

Therefore the equation of the tangent is $\quad Y - y = \frac{dy}{dx}(X - x)$

Normal at (x, y)

The normal at (x, y) being perpendicular to tangent will have its slope as $-1\Big/\frac{dy}{dx}$ and hence its equation is

$$Y - y = \frac{-1}{dy/dx}(X - x)$$

Geometrical meaning of dy/dx

dy/dx represents the slope of the tangent to the given curve $y = f(x)$ at any point (x, y)

$$\therefore \qquad \frac{dy}{dx} = \tan\psi$$

where ψ is the angle which the tangent to the curve makes with +ve direction of x-axis.

In case we are to find the tangent at any point (x_1, y_1) then $\left(\frac{dy}{dx}\right)_{(x_1, y_1)}$ *i.e.* the value of $\frac{dy}{dx}$ at (x_1, y_1) will represent the slope of the tangent and hence its equation in this case will be

$$y - y_1 = \left(\frac{dy}{dx}\right)_{(x_1, y_1)} (x - x_1)$$

Normal $$y - y_1 = \frac{-1}{(dy/dx)_{(x_1, y_1)}} (x - x_1)$$

Condition for tangent to be parallel or perpendicular to *x*-axis

If tangent is parallel to *x*-axis or normal is perpendicular to *x*-axis then

$$\frac{dy}{dx} = 0$$

If tangent is perpendicular to *x*-axis or normal is parallel to *x*-axis then

$$\frac{dy}{dx} = \infty \text{ or } \frac{dx}{dy} = 0 .$$

2.4 Maxima and Minima

For the function $y = f(x)$ at the maximum as well as minimum point the tangent is parallel to *x*-axis so that its slope is zero.

Calculate $\frac{dy}{dx} = 0$ and solve for *x*. Suppose one root of $\frac{dy}{dx} = 0$ is at *x*=*a*.

If $\frac{d^2y}{d^2x} = -ve$ for *x*=*a*, then maximum at *x*=*a*.

If $\frac{d^2y}{d^2x} = +ve$ for *x*=*a*, then minimum at *x*=*a*.

If $\frac{d^2y}{d^2x} = 0$ at *x*=*a*, then find $\frac{d^3y}{d^3x}$.

If $\frac{d^3y}{d^3x} \neq 0$ at *x*=*a*, neither maximum nor minimum at *x*=*a*.

If $\frac{d^3y}{d^3x} = 0$ at *x*=*a*, then find $\frac{d^4y}{d^4x}$.

If $\frac{d^4y}{d^4x} > 0$ i.e +*ve* at *x*=*a*, then *y* is minimum at *x*=*a* and if $\frac{d^4y}{d^4x} < 0$ i.e -*ve* at *x*=*a*, then *y* is maximum at *x*=*a* and so on.

2.5 Integration

Indefinite Integration

If $\frac{d}{dx}[F(x)+c]=f(x)$, then we say that $F(x)+c$ is an indefinite integral or antiderivative of $f(x)$ and we write

$$\int f(x)dx = F(x)+c$$

Some standard Integrals

1. $\int x^n dx = \frac{x^{n+1}}{n+1}+c \quad (n\neq -1)$

2. $\int \frac{1}{x^n}dx = \frac{-1}{(n-1)x^{n-1}}+c \quad (n\neq 1)$

3. $\int \frac{1}{\sqrt{x}}dx = 2\sqrt{x}+c$

4. $\int \frac{1}{x}dx = \log x + c$

5. $\int e^x dx = e^x + c$

6. $\int a^x dx = \frac{a^x}{\log a}+c$

7. $\int \cos x\, dx = \sin x + c$

8. $\int \sin x\, dx = -\cos x + c$

9. $\int \sec^2 x dx = \tan x + c$

10. $\int \operatorname{cosec}^2 x\, dx = -\cot x + c$

11. $\int \sec x \tan x\, dx = \sec x + c$

12. $\int \operatorname{cosec} x \cot x\, dx = -\operatorname{cosec} x + c$

13. $\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x + c$

14. $\int \frac{dx}{1+x^2} = \tan^{-1} x + c$

15. $\int \frac{dx}{x\sqrt{x^2-1}} = \sec^{-1} x + c$

16. $\int \cosh x\, dx = \sinh x + c$

17. $\int \sinh x\, dx = \cosh x + c$

18. $\int \operatorname{sech}^2 x dx = \tanh x + c$

19. $\int \operatorname{cosech}^2 x\, dx = -\coth x + c$

20. $\int \operatorname{sech} x \tanh x\, dx = -\operatorname{sech} x + c$

21. $\int \operatorname{cosech} x \coth x\, dx = -\operatorname{cosech} x + c$

22. $\int \tan x dx = \log(\sec x) + c$

23. $\int \cot x\, dx = \log(\sin x) + c$

24. $\int \sec x\, dx = \log(\sec x + \tan x) + c$

25. $\int \operatorname{cosec} x\, dx = \log\left[\tan\frac{x}{2}\right] + c$

26. $\int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + c$

27. $\int \frac{dx}{x^2 - a^2} = \frac{1}{2a}\log\left[\frac{x-a}{x+a}\right] + c;$ if $x > a$

28. $\int \frac{dx}{a^2 - x^2} = \frac{1}{2a}\log\left[\frac{a+x}{a-x}\right] + c;$ if $x < a$

29. $\int \frac{dx}{\sqrt{x^2 + a^2}} = \log\left[x + \sqrt{x^2 + a^2}\right] + c$ or $\sinh^{-1}\left(\frac{x}{a}\right) + c$

30. $\int \frac{dx}{\sqrt{x^2 - a^2}} = \log\left[x + \sqrt{x^2 - a^2}\right] + c$ or $\cosh^{-1}\left(\frac{x}{a}\right) + c$

31. $\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\frac{x}{a} + c$

32. $\int \sqrt{x^2 + a^2}\,dx = \frac{x}{2}\sqrt{x^2 + a^2} + \frac{a^2}{2}\log\left[x + \sqrt{x^2 + a^2}\right]$

33. $\int \sqrt{x^2 - a^2}\,dx = \frac{x}{2}\sqrt{x^2 - a^2} - \frac{a^2}{2}\log\left[x + \sqrt{x^2 - a^2}\right]$

34. $\int \sqrt{a^2 - x^2}\,dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + c$

35. $\int \frac{dx}{x\sqrt{x^2 - a^2}} = \frac{1}{a}\sec^{-1}\frac{x}{a} + c = -\frac{1}{a}\operatorname{cosec}^{-1}\frac{x}{a} + c$

36. $\int_0^a \sin\frac{m\pi x}{L}\sin\frac{n\pi x}{L}\,dx = \frac{a}{2}\delta_{mn}$

$= 0, \quad m \neq n$

$= \frac{a}{2}, \quad m = n$

Rules of Integration

1. $\int [f_1(x) + f_2(x)]dx = \int f_1(x)dx + \int f_2(x)dx$

2. $\int k \cdot f(x)dx = k\int f(x)dx$, where k is constant

3. $\int [k_1 f_1(x) + k_2 f_2(x)]dx = k_1\int f_1(x)dx + k_2\int f_2(x)dx$, where k_1 and k_2 are constants

Rule of integration by substitution

1. If $x = \phi(t)$, $\int f(x)dx = \int f(x)\frac{dx}{dt}dt = \int f[\phi(t)]\phi'(t)dt$

2. $\int f(ax+b)dx = \frac{g(ax+b)}{a} + c$

3. $\int [f(x)]^n f'(x)dx = \frac{[f(x)]^{n+1}}{n+1} + c : (n \neq -1)$

4. $\int \frac{f'(x)}{f(x)}dx = \log f(x) + c$

Rules of integration by partial fraction

This method can be used to evaluate an integral of the type $\int \frac{P(x)}{Q(x)}dx$

where (i) $P(x)$ & $Q(x)$ are Polynomials in x

(ii) Degree of $P(x)$ < degree of $Q(x)$

(iii) $Q(x)$ contains two/more distinct linear/quadratic factors i.e.

$$\frac{P(x)}{Q(x)} = \frac{A}{(a_1x+b_1)} + \frac{B}{(a_2x+b_2)} + \frac{C}{(a_3x+b_3)}$$

$$\int uvdx = u\int vdx - \int\left[\frac{du}{dx}\int vdx\right]dx$$

<u>2.5.1 Gamma integral</u>

(i) Gamma integral is given by $\Gamma(n) = \int_0^\infty x^{n-1}e^{-x}dx = \underline{|(n-1)}$.

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

(ii) $\int_0^\infty x^n e^{-Bx^2}dx = \frac{1}{2B^{\frac{n+1}{2}}}\Gamma\left(\frac{n+1}{2}\right)$

3. Differential Equations

Order and degree of a differential equation

The order of a differential equation is the order of the highest differential co-efficient present in the equation.

Example: $\frac{d^2y}{dx^2}+2\frac{dy}{dx}+3y=0$ is a second order differential equation.

The degree of a differential equation is the degree of the highest derivative after removing the radical sign and fraction.

Example: $\left(\frac{d^2y}{dx^2}\right)^2+2\left(\frac{dy}{dx}\right)^3+3y=0$ has degree of 3.

D.E. of the first order and first degree

1. **Separation of the variables:**

$$f(y)dy=\phi(x)dx$$

2. **Homogeneous Equation**

$\frac{dy}{dx}=\frac{f(x,y)}{\phi(x,y)}$ if each term of *f(x,y)* and $\phi(x,y)$ is of the same degree.

3. **Equations reducible to homogeneous form**

$$\frac{dy}{dx}=\frac{ax+by+c}{Ax+By+C},\ \text{let}\ \left.\begin{array}{l}x=X+h\\ y=Y+k\end{array}\right\}\Rightarrow\frac{dy}{dx}=\frac{dY}{dX}=\frac{aX+bY+ah+bk}{AX+BY+Ah+Bk}$$

Choose h, k so that $\left.\begin{array}{l}ah+bk+c=0\\ Ah+Bk+C=0\end{array}\right\}\Rightarrow\frac{dy}{dx}=\frac{aX+bY}{AX+BY}$

Case of failure: $\frac{a}{A}=\frac{b}{B}=\frac{1}{m}\quad\Rightarrow\quad\frac{dy}{dx}=\frac{ax+by+c}{m(ax+by)+C}$

4. **Linear Differential Equations**

$\frac{dy}{dx}+Py=Q$ where P and Q are function of x (but not y) or constant.

$$I.F.=e^{\int Pdx}\Rightarrow\qquad y\times I.F.=\int(Q\times I.F.)dx+c$$

5. **Equations reducible to the linear form**

$$\frac{dy}{dx}+Py=Qy^n \qquad \text{divide by } y^n \text{ and put } \frac{1}{y^{n-1}}=z$$

$$\Rightarrow \frac{1}{y^n}\frac{dy}{dx}+\frac{1}{y^{n-1}}P=Q \qquad \Rightarrow \frac{(1-n)}{y^n}\frac{dy}{dx}=\frac{dz}{dx}$$

$$\Rightarrow \frac{1}{1-n}\frac{dz}{dx}+Pz=Q$$

6. Exact differential Equation

$$Mdx+Ndy=0 \text{ if } \frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

$$\int_{y=\text{constant}} Mdx+\int(\text{terms of N not containing } x)dy=C$$

7. **Equations reducible to the exact form**

a) If $\dfrac{\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}}{N}$ is a function of x alone, say $f(x)$ then $\text{I.}F.=e^{\int f(x)dx}$ multiply with different equation.

b) If $\dfrac{\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}}{N}$ is a function of y alone, say $f(y)$ then $\text{I.}F.=e^{\int f(y)dy}$.

c) If M = $yf_1(xy)$ and N = $xf_2(xy)$, then $\text{I.}F.=\dfrac{1}{Mx-Ny}$

Linear D.E. of second order with constant coefficients

$\frac{d^2y}{dx^2}+P\frac{dy}{dx}+Qy=R$ where P, Q and R are function of x or constant.

$$y = \text{C.F.} + \text{P.I.}$$

C.F.

a) roots, real and different $\quad y=C_1e^{m_1x}+C_2e^{m_2x}$

b) roots, real and equal $\quad y=(C_1+C_2x)e^{m_2x}$

c) roots imaginary $\quad y=C_1e^{(\alpha+\alpha\beta)x}+C_2e^{(\alpha-\alpha\beta)x}$

$$=e^{\alpha x}\left[A\cos\beta x+B\sin\beta x\right]$$

P.I.

a) $\frac{1}{f(D)}e^{ax}=\frac{1}{f(a)}e^{ax}$ if $f(a)=0$ then $\frac{1}{f(D)}e^{ax}=x\cdot\frac{1}{f(a)}\cdot e^{ax}$

b) $\frac{1}{f(D)}x^n=[f(D)]^{-1}x^n$

c) $\frac{1}{f(D^2)}\sin ax=\frac{1}{f(-a^2)}\sin ax$ and $\frac{1}{f(D^2)}\cos ax=\frac{1}{f(-a^2)}\cos ax$

If $f(-a^2)=0$ then $\frac{1}{f(D^2)}\sin ax=\frac{1}{f(-a^2)}\sin ax$

d) $\frac{1}{f(D^2)}e^{ax}\phi(x)=e^{ax}\frac{1}{f(D+a)}\phi(x)$

e) $\frac{1}{D+a}\phi(x)=e^{-ax}\int e^{ax}\phi(x)dx$

f) $\frac{1}{f(D)}x^n\sin ax=\operatorname{Im}e^{ax}\frac{1}{f(D+a)}x^n$

4. Vectors

Cartesian coordinate system

Infinitesimal displacement $d\vec{l} = dx\hat{x} + dy\hat{y} + dz\hat{z}$

Volume element $d\tau = dxdydz$

Gradient: $\vec{\nabla} f = \frac{\partial f}{\partial x}\hat{x} + \frac{\partial f}{\partial y}\hat{y} + \frac{\partial f}{\partial z}\hat{z}$

Divergence: $\vec{\nabla}.\vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$

Curl: $\vec{\nabla} \times \vec{A} = \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\hat{x} + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\hat{y} + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\hat{z}$

Laplacian: $\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$

Spherical Polar Coordinate System(r, θ, ϕ)

$x = r\sin\theta\cos\phi, \quad y = r\sin\theta\sin\phi, \quad z = r\cos\theta$

$r = \sqrt{x^2 + y^2 + z^2}, \quad \theta = \cos^{-1}\frac{z}{r}, \quad \phi = \tan^{-1}\frac{y}{x}$

Infinitesimal displacement $d\vec{l} = dr\hat{r} + rd\theta\hat{\theta} + r\sin\theta d\phi\hat{\phi}$

Volume element $d\tau = r^2\sin\theta drd\theta d\phi$

r range from 0 to ∞, θ from 0 to π, and ϕ from 0 to 2π.

Gradient: $\vec{\nabla} f = \frac{\partial f}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial f}{\partial \theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}\hat{\phi}$

Divergence: $\vec{\nabla}.\vec{A} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 A_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta A_\theta\right) + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial \phi}$

Curl: $\vec{\nabla} \times \vec{A} = \frac{1}{r^2\sin\theta}\begin{vmatrix} \hat{r} & r\hat{\theta} & r\sin\theta\hat{\phi} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ A_r & rA_\theta & r\sin\theta A_\phi \end{vmatrix}$

Laplacian: $\nabla^2 f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\left(\frac{\partial^2 f}{\partial\phi^2}\right)$

Cylindrical Coordinate System (r, ϕ, z)

$x = r\cos\phi,\quad y = r\sin\phi,\quad z = z$ and $r = \sqrt{x^2 + y^2},\quad \phi = \tan^{-1}\frac{y}{x}$

Infinitesimal displacement $d\vec{l} = dr\hat{r} + rd\phi\hat{\phi} + dz\hat{z}$

Volume element $d\tau = rdrd\phi dz$

r range from 0 to ∞, ϕ from 0 to 2π, and z from $-\infty$ to $+\infty$.

Gradient: $\vec{\nabla} f = \frac{\partial f}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial f}{\partial\phi}\hat{\phi} + \frac{\partial f}{\partial z}\hat{z}$

Divergence: $\vec{\nabla}.\vec{A} = \frac{1}{r}\frac{\partial}{\partial r}(rA_r) + \frac{1}{r}\frac{\partial A_\phi}{\partial\phi} + \frac{\partial A_z}{\partial z}$

Curl: $\vec{\nabla}\times\vec{A} = \frac{1}{r}\begin{vmatrix} \hat{r} & r\hat{\phi} & \hat{z} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\phi} & \frac{\partial}{\partial z} \\ A_r & rA_\phi & A_z \end{vmatrix}$

Laplacian: $\nabla^2 f = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 f}{\partial\phi^2} + \frac{\partial^2 f}{\partial z^2}$

VECTOR IDENTITIES

Triple Product

(1) $\vec{A}.(\vec{B}\times\vec{C})=\vec{B}.(\vec{C}\times\vec{A})=\vec{C}.(\vec{A}\times\vec{B})$

(2) $\vec{A}\times(\vec{B}\times\vec{C})=\vec{B}(\vec{A}.\vec{C})-\vec{C}(\vec{A}.\vec{B})$

Product Rules

(3) $\vec{\nabla}(fg)=f(\vec{\nabla}g)+g(\vec{\nabla}f)$

(4) $\vec{\nabla}\left(\vec{A}.\vec{B}\right)=\vec{A}\times(\vec{\nabla}\times\vec{B})+\vec{B}\times(\vec{\nabla}\times\vec{A})+\left(\vec{A}.\vec{\nabla}\right)\vec{B}+(\vec{B}.\vec{\nabla})\vec{A}$

(5) $\vec{\nabla}.(f\vec{A})=f(\vec{\nabla}.\vec{A})+\vec{A}.(\vec{\nabla}f)$

(6) $\vec{\nabla}.(\vec{A}\times\vec{B})=\vec{B}.(\vec{\nabla}\times\vec{A})-\vec{A}.(\vec{\nabla}\times\vec{B})$

(7) $\vec{\nabla}\times(f\vec{A})=f(\vec{\nabla}\times\vec{A})-\vec{A}\times(\vec{\nabla}f)$

(8) $\vec{\nabla}\times\left(\vec{A}\times\vec{B}\right)=(\vec{B}.\vec{\nabla})\vec{A}-(\vec{A}.\vec{\nabla})\vec{B}+\vec{A}(\vec{\nabla}.\vec{B})-\vec{B}(\vec{\nabla}.\vec{A})$

Second Derivative

(9) $\vec{\nabla}.(\vec{\nabla}\times\vec{A})=0$ i.e. divergence of a curl is always zero.

(10) $\vec{\nabla}\times(\vec{\nabla}f)=0$ i.e. curl of a gradient is always zero.

(11) $\vec{\nabla}\times(\vec{\nabla}\times\vec{A})=\vec{\nabla}\left(\vec{\nabla}.\vec{A}\right)-\nabla^2\vec{A}$

FUNDAMENTAL THEOREMS

Gradient Theorem: $\int_a^b\left(\vec{\nabla}f\right).d\vec{l}=f(b)-f(a)$

Divergence Theorem: $\int\left(\vec{\nabla}.\vec{A}\right)d\tau=\oint\vec{A}.d\vec{a}$

Curl Theorem: $\int\left(\vec{\nabla}\times\vec{A}\right).d\vec{a}=\oint\vec{A}.d\vec{l}$

5. Algebra

5.1 Theory of Quadratic equations

1. Roots of the equation

$ax^2+bx+c=0$ are $x=\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$

Sum and Product of the roots

If α and β be the roots, then

$\alpha+\beta=-\dfrac{b}{a}$ and $\alpha.\beta=\dfrac{c}{a}$

2. To find the equation whose roots are α and β.

The required equation will be

$(x-\alpha)(x-\beta)=0$ or $x^2+(\alpha+\beta)x+\alpha.\beta=0$ or $x^2+Sx+P=0$

where S is the sum and P is the product of the root.

3. Nature of the roots.

Roots of the equation $ax^2+bx+c=0$ are $x=\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$.

The expression b^2-4ac is called **discriminant**.

(a) If $b^2-4ac\geq 0$, roots are **real**.

(i) If $b^2-4ac>0$, then roots are **real and unequal.**

(ii) If $b^2-4ac=0$, then roots are **real and equal** $\left(-\dfrac{b}{2a}\right)$.

(b) If $b^2-4ac<0$, then $\sqrt{b^2-4ac}$ is imaginary. Therefore roots are imaginary and unequal.

5.2 Logarithms

Properties of Logarithms ($a > 0, a \neq 1, m > 0, n > 0$)

1. $a^x = y$ then $x = \log_a y$
2. $\log_a a = 1$
3. $\log_a 1 = 0$
4. $\log_b a = \dfrac{1}{\log_a b}$ or $\log_b a.\log_a b = 1$
5. Base changing formula $\log_b a = \log_c a.\log_b c = \dfrac{\log_c a}{\log_c b}$
6. $\log_a mn = \log_a m + \log_a n$, $\quad \log_a \left(m/n\right) = \log_a m - \log_a n$
7. $\log_a m^n = n \log_a m$ Or in particular $\log_a a^n = n$
8. $\log_{a^q} n^q = \left(\dfrac{p}{q}\right) \log_a n$ Or in particular $\log_{n^q} n^q = \dfrac{p}{q}$
9. $a^{\log_a n} = n$

Rules of indices

1. $a^m \times a^n = a^{m+n}$
2. $\dfrac{a^m}{a^n} = a^{m-n}$
3. $\left(a^m\right)^n = a^{mn}$
4. $\left(a \times b\right)^m = a^m \times b^m$
5. $\left(\dfrac{a}{b}\right)^m = \dfrac{a^m}{b^m}$
6. $a^{-m} = \dfrac{1}{a^m}$
7. $a^0 = 1$

5.3 Permutations and Combinations

Permutation

Each of the different arrangements which can be made by taking some or all of a number of things is called permutation.

Combination

Each of the different groups or selections which can be made by taking some or all of a number of things (irrespective of the order) is called combination.

Fundamental Theorem

If there are m ways of doing a thing and for each of the m ways there are associated n ways of doing a second thing then the total number of ways of doing the two things will be mn.

Important Results

(a) Number of permutations of n dissimilar things taken r at a time.

$$^nP_r = \frac{n!}{(n-r)!} = n(n-1)(n-2)......(n-r+1)$$

where $n! = 1.2.3...............n$.

Note that $n! = n.(n-1)! = n.(n-1).(n-2)!$

(b) Number of permutations of n dissimilar things taken all at a time.

$$^nP_n = \frac{n!}{(n-r)!} = n(n-1)(n-2)......(n-n+1)$$

$$= n(n-1)(n-2)..........3.2.1 = n!$$

(c) Number of combinations of n dissimilar things taken r at a time.

$$^nC_r = \frac{n!}{(n-r)!r!} = \frac{^nP_r}{r!}$$

(d) Number of combinations of n dissimilar things taken all at a time.

$$^nC_n = \frac{n!}{(n-n)!n!} = \frac{1}{0!} = 1 \qquad \because 0! = 1$$

(e) If out of n things p are exactly alike of one kind, q exactly alike of second kind and r exactly alike of third kind and the rest all different, then the number of permutations of n things taken all at a time

$$= \frac{n!}{p!.q!.r!}$$

(f) If some or all of n things be taken at a time then the number of combinations will be

$$2^n - 1 \qquad \because \qquad {}^nC_1 + {}^nC_2 + ... + {}^nC_n = 2^n - 1$$

(g) ${}^nC_r = {}^nC_{n-r}$

(h) ${}^nC_{r_1} = {}^nC_{r_2} \Rightarrow r_1 = r_2$ or $r_1 + r_2 = n$.

(i) ${}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$

5.4 Binomial Theorem

(a) Statement of binomial theorem for positive and negative integral index

$$(x+a)^n = x^n + {}^nC_1 x^{n-1}a^1 + {}^nC_2 x^{n-2}a^2 + \ldots\ldots\ldots {}^nC_r x^{n-r}a^r + \ldots\ldots {}^nC_{n-1} x a^{n-1} + {}^nC_n a^n$$

$$(x-a)^n = x^n - {}^nC_1 x^{n-1}a^1 + {}^nC_2 x^{n-2}a^2 + \ldots\ldots\ldots {}^nC_r x^{n-r}\left(-a^r\right) + \ldots\ldots\ldots\ldots\ldots\ldots$$

(b) Number of terms and middle term

The number of terms in the expansion of $(x+a)^n$ is $n+1$.

If n **is even** there will be only one middle term *i.e.* $\left(\frac{n}{2}+1\right)th$.

If n **is odd** there will be two middle terms *i.e.* $\left(\frac{n+1}{2}\right)th$ and $\left(\frac{n+3}{2}\right)th$.

Expansion

1. $(a+b)^2 = a^2 + 2ab + b^2$
2. $(a-b)^2 = a^2 - 2ab + b^2$
3. $(a+b)^3 = a^3 + 3ab(a-b) - b^3$
4. $(a-b)^3 = a^3 - 3ab(a-b) - b^2$

Factorization

1. $a^2 - b^2 = (a+b)(a+b)$
2. $a^3 - b^3 = (a-b)(a + ab + b^2)$
3. $a^3 + b^3 = (a+b)(a^2 - ab + b^2)$
4. $a^4 - b^4 = (a-b)(a+b)(a + b^2)$
5. $a^n - b^n = (a-b)(a^{n-1} + a^{n-2}b + a^{n-3}b^2 + \ldots.)$
6. $a^n + b^n = (a+b)(a^{n-1} - a^{n-2}b + a^{n-3}b^2 + \ldots.)$

Sterling's formula

Using summation notation, binomial expansion can be written as

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

Sterling's approximation (or **Sterling's formula**) is an approximation for large factorials.

$$\ln\left(\underline{|n}\right) = n\ln n - n \quad \text{where } n \text{ is very large}$$

5.5 Determinants

In linear algebra the **determinant** is a value associated with a square matrix. The determinant of a matrix A is denoted by $\det(A)$, or $|A|$. For instance, the determinant of the matrix

If $A=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad-bc$

If $A=\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$ then $\det(A)=\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$

Properties

(a) The values of determinant is not altered by changing rows into columns and columns into rows.

e.g. $$\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{vmatrix} = \begin{vmatrix} 1 & x & x^2 \\ 1 & y & z \\ 1 & z & z^2 \end{vmatrix}$$

(b) If any two adjacent rows or two adjacent columns of a determinant are interchanged the determinant retains its absolute value but changes its sign.

e.g. $$\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{vmatrix} = -\begin{vmatrix} x & y & z \\ 1 & 1 & 1 \\ x^2 & y^2 & z^2 \end{vmatrix}$$

(c) If any two rows or two columns of determinant are identical then the determinant vanishes. Thus

$$\begin{vmatrix} a_1 & c_1 & c_1 \\ a_2 & c_2 & c_2 \\ a_3 & c_3 & c_3 \end{vmatrix}$$

(d) If each constituent in any row or in any column be multiplied by the same factor then the determinant is multiplied by that factor

$$\begin{vmatrix} pa_1 & b_1 & c_1 \\ pa_2 & b_2 & c_2 \\ pa_3 & b_3 & c_3 \end{vmatrix} = p\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \text{ and } \begin{vmatrix} a_1 & b_1 & c_1 \\ qa_2 & qb_2 & qc_2 \\ ra_3 & rb_3 & rc_3 \end{vmatrix} = qr\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

(e) If each constituent in any row or in any column consists of r terms then the determinant can be expressed as the sum of r determinants.

Thus $$\begin{vmatrix} a_1+\alpha_1 & b_1 & c_1 \\ a_2+\alpha_2 & b_2 & c_2 \\ a_3+\alpha_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} \alpha_1 & b_1 & c_1 \\ \alpha_2 & b_2 & c_2 \\ \alpha_3 & b_3 & c_3 \end{vmatrix}$$

(f) If one row or column is k times the other row or columns respectively then determinant of matrix will be 0.

e.g. $$\begin{vmatrix} a & k.a & c \\ d & k.d & f \\ g & k.g & i \end{vmatrix} = 0 \text{ and } \begin{vmatrix} a & b & c \\ k.a & k.b & k.c \\ g & h & i \end{vmatrix} = 0$$

Some basic properties of determinants are:

1. $\det(I_n) = I$ where I_n is the $n \times n$ identity matrix.

2. $\det(A^T) = \det(A)$ where A^T is transpose of A.

3. $\det(A^{-1}) = \dfrac{1}{\det(A)}$ where A^{-1} is inverse of A.

4. For square matrices A and B of equal size,

$$\det(AB) = \det(A)\det(B)$$

5. $\det(cA) = c^n \det(A)$ for an $n \times n$ matrix

6. Conic Section

In the Cartesian coordinate system the graph of a quadratic equation of two variables represent a conic section which is given by $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$.

The conic sections described by this equation can be classified with the **discriminant** $D = B^2 - 4AC$

- if $D < 0$, the equation represents an ellipse
- if $D < 0$, $A = C$ and $B = 0$, the equation represents a circle which is a special case of an ellipse;
- if $D = 0$, the equation represents a parabola
- if $D > 0$ the equation represents a hyperbola
- if we also have $D > 0$, $A + C = 0$, the equation represents a rectangular hyperbola

Note that A and B are polynomial coefficients, not the lengths of semi-major/minor axis as defined in some sources.

Conic section	Equation	Eccentricity	Semi-lactus rectum	Polar equation	Parametric form
Circle	$x^2 + y^2 = a^2$	0	a	$r = a$	$x = a\cos\theta, y = a\sin\theta$
Ellipse	$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$	$e = \sqrt{1 - \frac{b^2}{a^2}}$	$l = \frac{b^2}{a}$	$\frac{l}{r} = 1 + e\cos\theta$ $0 < e < 1$	$x = a\cos\theta, y = b\sin\theta$
Parabola	$y^2 = 4ax$	$e = 1$	$2a$	$\frac{l}{r} = 1 + \cos\theta$	$x = at^2, y = 2at$
Hyperbola	$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$	$e = \sqrt{1 + \frac{b^2}{a^2}}$	$l = \frac{b^2}{a}$	$\frac{l}{r} = 1 + e\cos\theta$ $e > 1$	$x = a\tan\theta, y = b\sec\theta$

7. Probability

Probability

The probability p_r of occurrence of an event *r* in a system is defined with respect to statistical ensemble of *N* such a systems. If N_r systems in the ensemble exhibit the event *r* then

$$p_r = \frac{N_r}{N}$$

Probability density

The probability density $\rho(u)$ is defined by the property that $\rho(u)du$ yields the probability of finding the continuous variable u in the range between u and $u + du$.

Mean value

The mean value of u is denoted by $\langle u\rangle$ as defined as $\langle u\rangle = \sum_r p_r u_r$ where the sum is over all possible value values u_r of the variable u and p_r is denotes the probability of occurrence of the particular value u_r .Above definition is for discrete variable .

For continuous variable u, $\langle u\rangle = \int u\rho(u)du$

Dispersions or variance

The dispersion of u is defined as $\sigma^2 = \langle(\Delta u)^2\rangle = \sum_r p_r(u_r - \langle u\rangle)^2$ which is equivalent to $\sigma^2 = \langle(\Delta u)^2\rangle = \sum_r (\langle u^2\rangle - \langle u\rangle^2)$

Joint probability

If both events A and B occur on a single performance of an experiment, this is called the intersection or joint probability of A and B, denoted as $p(A \cap B)$.

Independent probability

If two events, A and B are independent then the joint probability is

$$p(A \cap B) = p(A).P(B)$$

Mutually exclusive

If either event A or event B or both events occur on a single performance of an experiment this is called the union of the events A and B denoted as $p(A \cup B)$. If two events are mutually exclusive then the probability of either occurring is

$$p(A \cup B) = p(A) + P(B)$$

Not mutually exclusive

If the events are not mutually exclusive then

$$p(A \cup B) = p(A) + P(B) - p(A \cap B)$$

Conditional probability

Conditional probability is the probability of some event A, given the occurrence of some other event B. Conditional probability is written $p(A/B)$, and is read "the probability of A, given B". It is defined by

$$p(A/B) = \frac{p(A \cap B)}{p(B)}$$