(a) Separable Differential Equations

If a differential equation can be written in the form

$$f(y)\,dy = \phi(x)\,dx$$

We say that variables are separable, y on the left hand side and x on the right hand side. We get the solution by integrating both sides.

Example: Solve the differential equation $9yy' + 4x = 0$.

Solution: By separating variables we have

$$9y\,dy = -4x\,dx$$

Integrating both sides, we get

$$\frac{9}{2}y^2 = -2x^2 + c' \Rightarrow \frac{x^2}{9} + \frac{y^2}{4} = c, \text{ where } c = \frac{c'}{18}.$$

The solution represents a family of ellipses.

Example: Solve the differential equation $y' = 1 + y^2$.

Solution: By separating variables we have

$$\frac{dy}{1+y^2} = dx$$

Integrating both sides, we get

$$\tan^{-1} y = x + c \Rightarrow y = \tan(x+c)$$

Example: Solve the differential equation $(yx^2 + y)dy + (xy^2 + x)dx = 0$.

Solution: $(yx^2 + y)\frac{dy}{dx} + (xy^2 + x) = 0 \Rightarrow \frac{y\,dy}{y^2+1} = -\frac{x}{x^2+1}dx$

$$\Rightarrow \frac{1}{2}\log(y^2+1) = -\frac{1}{2}\log(x^2+1) + c' \Rightarrow (y^2+1)(x^2+1) = c$$

Example: Solve the differential equation $y' = -\frac{y}{x}$, with $y(1) = 1$.

Solution: By separating variables we have

$$\frac{dy}{y} = -\frac{dx}{x}$$

Integrating both sides, we get

$$\ln|y| = -\ln|x| + \ln|c| \Rightarrow y = \frac{c}{x}$$

Example: Solve the differential equation $y' = -2xy$, with $y(0) = 1$.

Solution: By separating variables we have

$$\frac{dy}{y} = -2xdx$$

Integrating both sides, we get

$$\ln|y| = -x^2 + c' \Rightarrow |y| = e^{-x^2+c'}$$

Setting $e^{c'} = +c$ when $y > 0$, and $e^{c'} = -c$ when $y < 0$, and admitting also $c = 0$ (which gives the solution $y = 0$), we get the general solution

$$y = ce^{-x^2}$$

$\because y(0) = 1 \Rightarrow c = 1 \Rightarrow y = e^{-x^2}$

Example: Solve the differential equation $x\frac{dy}{dx} + \cot y = 0$, given $y = \frac{\pi}{4}$ where $x = \sqrt{2}$.

Solution: $x\frac{dy}{dx} + \cot y = 0 \Rightarrow xdy = -\cot y\, dx \Rightarrow \tan y\, dy = -\frac{dx}{x}$

$\Rightarrow \int \tan y \quad dy = -\int \frac{dx}{x} + \log c \Rightarrow \log \sec y = -\log x + \log c \Rightarrow \log \sec y + \log x = \log c$

$\Rightarrow \log \frac{x}{\cos y} = \log c \Rightarrow x = c \cos y$

On putting $y = \frac{\pi}{4}$ and $x = \sqrt{2}$ we get $\sqrt{2} = c \cos \frac{\pi}{4}$ $\Rightarrow c = 2$ $\Rightarrow x = 2\cos y$.