1. Ordinary Differential Equations

The order of a differential equation is the order of the highest derivative that appears in the equation.

For example,

$y' = \cos x$ (First order differential equation),

$y'' + 4y = 0$ (Second order differential equation),

$x^2 y''' y' + 2y'' = x^2 y^2$ (Third order differential equation)

The first-order differential equations contains only y' and may contain y and given functions of x. Hence we can write

$$F(x, y, y') = 0 \text{ or } y' = f(x, y) \qquad \dots(1)$$

Concept of Solution

A **solution** of a given first-order differential equation (1) on some open interval $a < x < b$ is a function $y = h(x)$ that has derivative $y' = h'(x)$ and satisfies (1) for all x in that interval. Thus equation (1) becomes an identity if we replace the unknown function y by $h(x)$ and y' by $h'(x)$.

Example: Verify that $y = x^2$ is a solution of $xy' = 2y$ for all x.

Substitute $y = x^2$ and $y' = 2x$ into $xy' = x \times 2x = 2x^2 = 2y$, an identity in x.

Implicit Solution:

Sometimes a solution of differential equation will appear as an implicit function, i.e.

$$H(x, y) = 0$$

and is called an implicit solution, in contrast to an explicit solution $y = h(x)$.

Example: The function y of x implicitly given by $x^2 + y^2 - 1 = 0 \quad (y > 0)$, represents a semicircle of unit radius in the upper-half plane. This function is an implicit solution of the differential equation $yy' = -x$ on the interval $-1 < x < 1$.

General and Particular Solution:

Consider the differential equation $y' = \cos x$.

Its solution will be $y = \sin x + c$ where c is an arbitrary constant. Such a function involving an arbitrary constant is called a **general solution** of a first order differential equation.

If we choose specific c ($c = 2$ or 0 or $-5/3$, etc), we obtain what is called a **particular solution** of that equation.

Thus $y = \sin x + c$ is a general solution of $y' = \cos x$, and $y = \sin x$, $y = \sin x - 2$, $y = \sin x + 0.75$, etc. are particular solutions.

Singular Solution:

A differential equation may sometimes have an additional solution that can not be obtained from the general solution and is then called a singular solution.

For example, $y'^2 - xy' + y = 0$ has the general solution $y = cx - c^2$. Substitution also shows that the $y = \frac{x^2}{4}$ is also a solution. This is a singular solution because we cannot obtain it from $y = cx - c^2$ by choosing a suitable c.

Initial Value Problems

A differential equation together with an initial condition is called an **initial value problem.** It is of the form

$$y' = f(x, y), \qquad y(x_0) = y_0$$

The initial condition $y(x_0) = y_0$ is used to determine a value of c in the general solution.