(f) Nonhomogeneous Equation of the form $x^2y''+axy'+by=x^m$

Consider linear differential equations in y, xy' and x^2y'' that is

$$x^2y''+axy'+by=x^m$$

Substitute $x=e^z$, then $dx=e^z dz \Rightarrow \frac{d}{dx}=\frac{1}{e^z}\frac{d}{dz} \Rightarrow e^z\frac{d}{dx}=\frac{d}{dz} \Rightarrow x\frac{d}{dx}=\frac{d}{dz}=D$

$$\frac{d^2}{dx^2}=\frac{d}{dx}\left(\frac{d}{dx}\right)=\frac{1}{e^z}\frac{d}{dz}\left(\frac{1}{e^z}\frac{d}{dz}\right)=\frac{1}{e^z}\left(-\frac{1}{e^z}\frac{d}{dz}+\frac{1}{e^z}\frac{d^2}{dz^2}\right)=\frac{1}{e^{2z}}\left(-\frac{d}{dz}+\frac{d^2}{dz^2}\right)=\frac{1}{x^2}\left(D^2-D\right)$$

$$\Rightarrow x^2\frac{d^2}{dx^2}=D(D-1)$$

Similarly $x^3\frac{d^3}{dx^3}=D(D-1)(D-2)$

Thus $x^2y''+axy'+by=x^m \Rightarrow \left[D(D-1)+aD+b\right]y=0 \Rightarrow \left[D^2+(a-1)D+b\right]y=e^{mz}$

$$\Rightarrow y''+(a-1)y'+by=e^{mz} \text{ where } y'=\frac{dy}{dz}.$$

This is second order nonhomogeneous equation with constant coefficients.

Example: Solve $x^2y''-2xy'-4y=x^4$.

Solution: Compare with $x^2y''+axy'+by=x^m$, then $a=-2$ and $b=-4$

Substitute $x=e^z \Rightarrow y''+(a-1)y'+by=e^{mz}$ where $y'=\frac{dy}{dz} \Rightarrow y''-3y'-4y=e^{mz}$

The **characteristic equation is**

$$\lambda^2-3\lambda-4=0 \Rightarrow \lambda=-1,\ 4$$

Thus solution is $y_h(z)=c_1e^{-z}+c_2e^{4z} \Rightarrow y_h(x)=c_1x^{-1}+c_2x^4$

$$y_p(z)=\frac{1}{D^2-3D+4}e^{4z}=z\frac{1}{2(4)-3}e^{4z}=\frac{ze^{4z}}{5} \Rightarrow y_p(x)=\frac{1}{5}x^4\ln x$$

Thus general solution is

$$y=y_h(x)+y_p(x)=c_1x^{-1}+c_2x^4+\frac{1}{5}x^4\ln x$$