(g) Singular Points of Linear Differential Equations of Second Order

When a linear homogeneous second-order ODE is written in the form

$$y'' + p(x)y' + q(x)y = 0 \qquad \ldots\ldots(1)$$

points x_0 for which $p(x)$ and $q(x)$ are finite are termed **ordinary points** of the ODE.

However, if either $p(x)$ or $q(x)$ diverges as $x \to x_0$, the point x_0 is called a **singular point.** Singular points are further classified as **regular** or **irregular** (or essential singularities):

- A singular point x_0 is **regular** if either $p(x)$ or $q(x)$ diverges there, but $(x-x_0)p(x)$ and $(x-x_0)^2 q(x)$ remain finite.
- A singular point x_0 is **irregular** if $p(x)$ diverges faster than $\frac{1}{(x-x_0)}$ so that $(x-x_0)p(x)$ goes to infinity as $x \to x_0$ or if $q(x)$ diverges faster than $\frac{1}{(x-x_0)^2}$ so that $(x-x_0)^2 q(x)$ goes to infinity as $x \to x_0$.

These definitions hold for all finite values of x_0.

To analyze the behavior at $x \to \infty$, we set $x = \frac{1}{z}$, substitute into the differential equation, and examine the behavior in the limit $z \to 0$.

The equation (1) is originally in the dependent variable $y(x)$, will now be written in terms of $w(z)$ where $w(z) = y(z^{-1})$. Thus

$$y' = \frac{dy(x)}{dx} = \frac{dy(z^{-1})}{dz}\frac{dz}{dx} = \frac{dw(z)}{dz}\left(-\frac{1}{x^2}\right) = -z^2 w' \qquad \ldots\ldots(2)$$

$$y'' = \frac{dy'}{dx} = \frac{dy'}{dz}\frac{dz}{dx} = \left(-\frac{1}{x^2}\right)\frac{d}{dz}\left[-z^2 w'\right] = \left(-z^2\right)\frac{d}{dz}\left[-z^2 w'\right] = z^4 w'' + 2z^3 w' \qquad \ldots\ldots(3)$$

Substituting y' and y'' in equation (1), we get

$$z^4 w'' + \left[2z^3 - z^2 p(z^{-1})\right]w' + q(z^{-1})w = 0$$

$$\Rightarrow w''+\left[\frac{2z-p\left(z^{-1}\right)}{z^2}\right]w'+\frac{q\left(z^{-1}\right)}{z^4}w=0 \qquad \text{.......(4)}$$

From equation (4), we see that the possibility of a singularity at $z=0$ depends on the behavior of

$$\frac{2z-p\left(z^{-1}\right)}{z^2} \text{ and } \frac{q\left(z^{-1}\right)}{z^4}.$$

If these two expressions remain finite at $z=0$, the point $x=\infty$ is an ordinary point. If they diverge no more rapidly than $\frac{1}{z}$ and $\frac{1}{z^2}$, respectively, $x=\infty$ is a regular singular point; otherwise it is an irregular singular point (or essential singularity).

Example: Consider Bessel's equation $x^2y''+xy'+\left(x^2-n^2\right)y=0$.

$$\Rightarrow y''+\frac{1}{x}y'+\left(1-\frac{n^2}{x^2}\right)y=0 \Rightarrow p(x)=\frac{1}{x},\quad q(x)=\left(1-\frac{n^2}{x^2}\right)\Rightarrow\left(x^2-n^2\right)=\text{finite}$$

which shows that the point $x=0$ is a regular singularity.

$$\left(x-x_0\right)p(x)\Rightarrow(x-0)\frac{1}{x}=1 \text{ and } \left(x-x_0\right)^2q(x)=(x-0)^2\left(1-\frac{n^2}{x^2}\right)$$

By inspection we see that there are no other singularities in the finite range.

Let us check at $x\to\infty\,(z\to 0)$ for coefficient

$$\frac{2z-p\left(z^{-1}\right)}{z^2}=\frac{2z-z}{z^2} \text{ and } \frac{q\left(z^{-1}\right)}{z^2}=\frac{1-n^2z^2}{z^4}.$$

Since the latter expression diverges as $\frac{1}{z^4}$, the point $x=\infty$ is an irregular or essential singularity.

Singularities of Some Important ODEs

S.N.	Equation	Regular Singularity $x =$	Irregular Singularity $x =$
1	Hypergeometric $x(x-1)y'' + [(1+a+b)x + c]y' + aby = 0$	0, 1, ∞	......
2	Legendre $(1-x^2)y'' - 2xy' + l(l+1)y = 0$	-1, 1, ∞	......
3	Chebyshev $(1-x^2)y'' - xy' + n^2 y = 0$	-1, 1, ∞	......
4	Confluence hypergeometric $xy'' + (c-x)y' - ay = 0$	0	∞
5	Bessel $x^2 y'' + xy' + (x^2 - n^2)y = 0$.	0	∞
6	Laguerre $xy'' + (1-x)y' + ay = 0$	0	∞
7	Simple Harmonic oscillator $y'' + \omega^2 y = 0$	......	∞
8	Hermite $y'' - 2xy' + 2\alpha y = 0$	......	∞