(e) Planck's Theory of Radiation (Quantum Mechanical Theory)

Basic Assumption and Methodology

(1) Max Planck used Boltzmann distribution to determine the average energy of black body radiation. The probability to find energy between E to $E+dE$ is $P(E)dE \propto \frac{1}{k_BT}\exp-\left(\frac{E}{k_BT}\right)dE$. The average energy is given by $\langle E\rangle = \frac{\int_0^\infty E\,P(E)dE}{\int_0^\infty P(E)dE}$.

(2) Planck assumed that energy E is not continuous rather discrete values. According to Planck, the allowed energy for black body radiation for frequency ν is $E = nh\nu$ where $n = 0,1,2,3...$

(3) The average energy is $\langle E\rangle = \frac{\sum_{n=0} EP(E)}{\sum_{n=0} P(E)} = \frac{\sum_{n=0}\frac{nh\nu}{k_BT}\exp\left(-\frac{nh\nu}{k_BT}\right)}{\sum_{n=0}\frac{1}{k_BT}\exp\left(-\frac{nh\nu}{k_BT}\right)} = \frac{h\nu}{\exp\left(\frac{h\nu}{k_BT}\right)-1}$

Derivation of Average Energy

Put the value, $\alpha = \frac{h\nu}{kT}$, $\quad \langle E\rangle = k_BT\frac{\sum_0^\infty n\alpha\exp(-n\alpha)}{\sum_0^\infty \exp(-n\alpha)}$

$$-\alpha\frac{d}{d\alpha}\ln\sum_{n=0}^{\infty}\exp(-n\alpha) = \frac{-\alpha\frac{d}{d\alpha}\sum_{n=0}^{\infty}\exp(-n\alpha)}{\sum_{n=0}^{\infty}\exp(-n\alpha)} = \frac{-\sum_{n=0}^{\infty}\alpha\frac{d}{d\alpha}\exp(-n\alpha)}{\sum_{n=0}^{\infty}\exp(-n\alpha)} = \frac{\sum_{n=0}^{\infty}n\alpha\exp(-n\alpha)}{\sum_{n=0}^{\infty}\exp(-n\alpha)}$$

$$\langle E\rangle = k_BT\left(-\alpha\frac{d}{d\alpha}\ln\sum_{n=0}^{\infty}\exp(-n\alpha)\right) = -h\nu\frac{d}{d\alpha}\ln\sum_0^\infty\exp(-n\alpha)$$

$$\sum_{n=0}^{\infty}\exp(-n\alpha) = 1+\exp(-\alpha)+\exp(-2\alpha)+\exp(-3\alpha)... = 1+X+X^2+X^3+...$$

where $X = \exp(-\alpha)$

$$1+X+X^2+X^3+... = (1-X)^{-1}$$

$$\langle E\rangle = -h\nu\frac{d}{d\alpha}\ln\left(1-\exp(-\alpha)\right)^{-1}$$

$$\frac{-h\nu}{\left(1-\exp(-\alpha)\right)^{-1}}(-1)\left(1-\exp(-\alpha)\right)^{-2}\exp(-\alpha) = \frac{h\nu}{\exp\left(\frac{h\nu}{k_B T}\right)-1}$$

(3) Planck also used the same analogy of Rayleigh and Jeans theory to count the number of standing waves $dN(\nu)$ in the frequency interval ν to $\nu + d\nu$ $dNd\nu = \frac{8\pi V a^3 \nu^2}{c^3}d\nu$

(4) Energy density $\rho(\nu)d\nu$ between frequency ν to $\nu + d\nu$ is $\rho(\nu)d\nu = N(\nu)d\nu\,\langle E\rangle$

$$\rho(\nu)d\nu = \frac{8\pi\nu^2}{c^3}.\frac{h\nu}{\exp\left(\frac{h\nu}{k_B T}\right)-1}$$

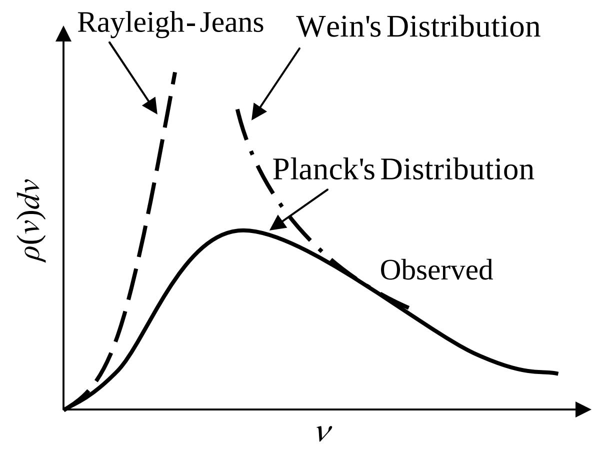

Figure 1: Energy density versus frequency according to Planck

The equation does accurately describe the low frequency (high wavelength) spectrum of thermal emission from objects, as well as accurately fit the experimental data for high frequency (short wavelengths) emission.