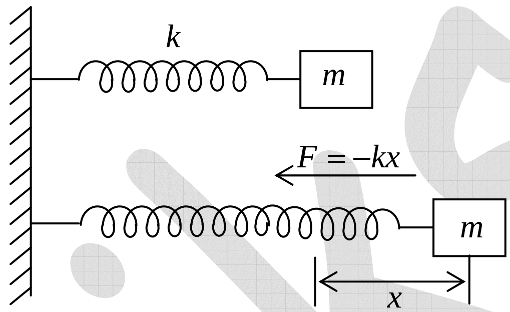
(a) Description of Simple Harmonic Motion

Equations of Motion (Hooke's Law)

Consider a mass m is attached to the spring, and the other end of the spring is connected to a rigid support such as a wall. If the system is left at rest at the equilibrium position $(x = 0)$ then there is no net force acting on the mass. However, if the mass is displaced from the equilibrium position, a restoring elastic force opposite to the displacement is exerted by the spring.



The equations of motion for a mass on a spring, using Hooke's law is

$$F = -kx$$

where F is the restoring elastic force exerted by the spring, k is the spring constant and x is the displacement from the equilibrium position.

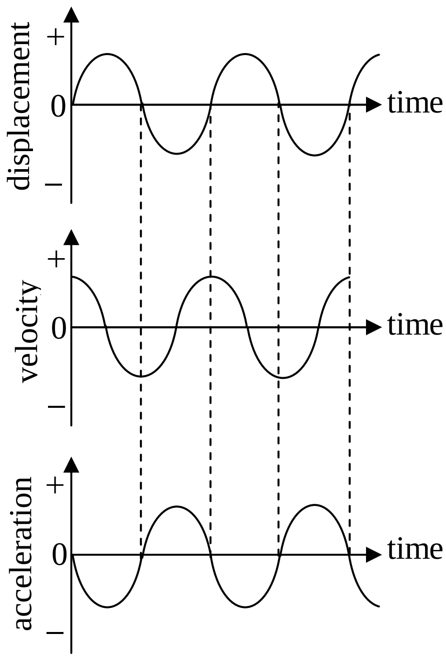
The equation of motion is a second-order linear ordinary differential equation obtained by means of Newton's second law.

$$F = m\frac{d^2x}{dt^2} = -kx \Rightarrow \frac{d^2x}{dt^2} + \frac{k}{m}x = 0$$

$\Rightarrow \dfrac{d^2x}{dt^2} + \omega^2 x = 0$ where $\omega = \sqrt{\dfrac{k}{m}}$ is the angular frequency of the oscillator.

We see from this equation that the higher the spring constant k, the stiffer the spring, and the greater the angular frequency of oscillation. A smaller mass will also increase the angular frequency for a particular spring.

Solving the differential equation above, a solution which is a sinusoidal function is obtained.



$$x(t) = Ce^{i\omega t} + De^{i\omega t} = A\sin(\omega t + \phi)$$

The velocity and acceleration is

$v(t) = \dfrac{dx}{dt} = A\omega\cos(\omega t + \phi) = \omega\sqrt{A^2 - x^2}$ and $a(t) = \dfrac{d^2x}{dt^2} = -A\omega^2\sin(\omega t + \phi) = -\omega^2 x$

Then, since $\omega = 2\pi f$, $f = \dfrac{1}{2\pi}\sqrt{\dfrac{k}{m}}$

and since $T = \dfrac{1}{f}$ where T is the time period, $T = 2\pi\sqrt{\dfrac{m}{k}}$

Energy of Simple Harmonic Motion

As an object vibrates in harmonic motion, energy is transferred between potential energy and kinetic energy. If we stretch a spring from its equilibrium (unstretched) position to a certain displacement the work done is equal to the stored potential energy in the spring. If we release the mass and allow it to begin moving back toward the equilibrium position, the potential energy begins changing into kinetic energy. As the mass passes through the equilibrium position, all of the potential energy has been converted into kinetic energy, and the speed of the mass is maximum. The kinetic energy in turn begins changing into potential energy, until all of the kinetic energy is converted into potential energy at maximum compression. The compressed spring then accelerates the mass back through the equilibrium to the original starting position, and the entire process repeats itself. The total energy of the system remains constant, that is,

Total Energy = Potential Energy + Kinetic Energy = a constant

The kinetic energy K of the system at time t is

$$K = \frac{1}{2}mv^2 = \frac{1}{2}m\omega^2A^2\cos^2(\omega t + \phi) = \frac{1}{2}kA^2\cos^2(\omega t + \phi) = \frac{1}{2}k\left(A^2 - x^2\right)$$

The potential energy is $U = \dfrac{1}{2}kx^2 = \dfrac{1}{2}kA^2\sin^2(\omega t + \phi)$

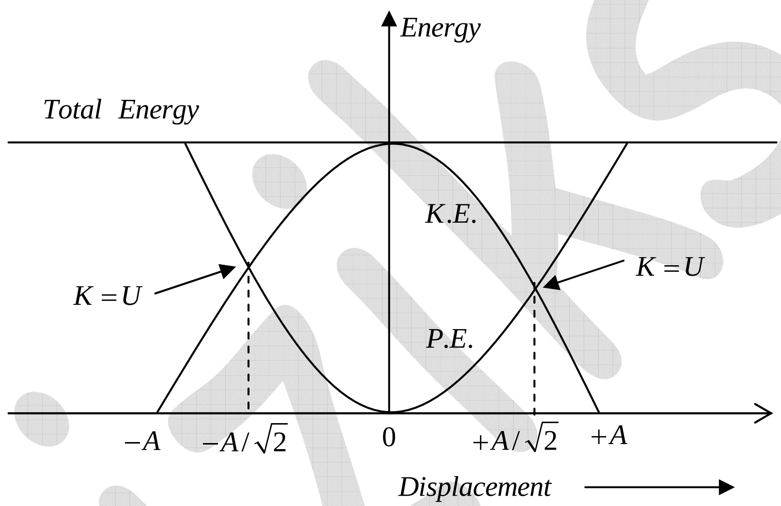
The total mechanical energy of the system therefore has the constant value

$$E = K + U = \frac{1}{2}kA^2$$

The total energy is constant and has the value is $\frac{1}{2}kA^2$. This is also the maximum value of the potential or the kinetic energy. Thus, at the time when P.E. is zero the K.E. is maximum or vice versa.

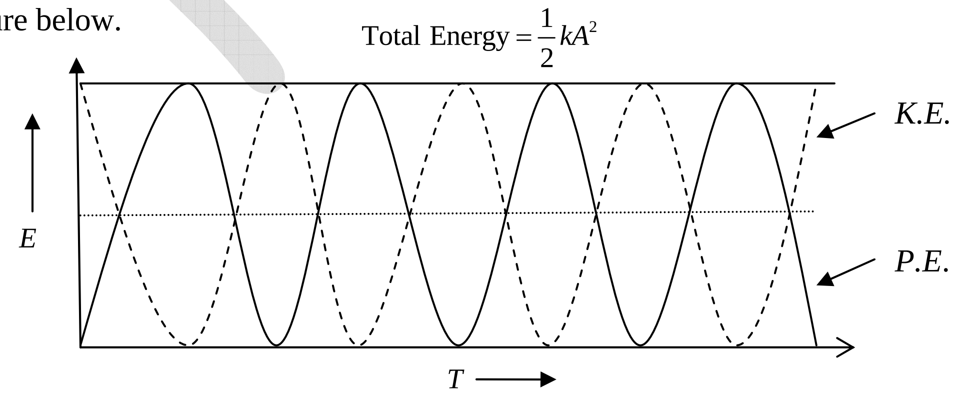
Energy vs Position Graph

The distribution of energy versus displacement for S.H.M. is shown in the figure. The P.E. is a parabola with vertex at $x = 0$, so the energy is stored in the oscillator both when x is positive and negative. The K.E. curve is an inverted parabola.



Energy vs Time Graph

The distribution of energy versus time for S.H.M. is shown in the figure. The P.E. $U = \frac{1}{2}kA^2\sin^2(\omega t + \phi)$ and K.E. is $K = \frac{1}{2}kA^2\cos^2(\omega t + \phi)$ variation with time is shown in figure below.



Frequency of energy change in S.H.M.

The energy of the particle in S.H.M. changes periodically from P.E. to K.E. Both U and K depends upon the square of the displacement x, i.e. each acquires same value twice in each oscillation. Thus the frequency of energy change is twice the frequency of oscillation.

Time Average of Kinetic and Potential Energy

Time average of kinetic energy $K(t)_{av} = \frac{\int_0^T K(t)\,dt}{T} = \frac{\int_0^T \frac{1}{2}kA^2\cos^2(\omega t+\phi)\,dt}{T} = \frac{1}{4}kA^2$

Time average of potential energy $U(t)_{av} = \frac{\int_0^T U(t)\,dt}{T} = \frac{\int_0^T \frac{1}{2}kA^2\sin^2(\omega t+\phi)\,dt}{T} = \frac{1}{4}kA^2$

Thus the time average of kinetic energy is equal to the time average of the potential energy which is half the total energy.

Position Average of Kinetic and Potential Energy

Position average of kinetic energy $K(x)_{av} = \frac{\int_0^A K(x)\,dt}{A} = \frac{\int_0^A \frac{1}{2}k\left(A^2-x^2\right)}{A} = \frac{1}{3}kA^2$

Position average of potential energy $U(x)_{av} = \frac{\int_0^T U(x)\,dt}{A} = \frac{\int_0^A \frac{1}{2}kx^2\,dx}{A} = \frac{1}{6}kA^2$

Thus the position average of kinetic energy is equal to the twice of the position average of the potential energy.