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Learn Physics in Right Way

JEST Physics-2022

Solution

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Part-A: 1-Mark Questions

Q2. The probability that you get a sum m from a throw of two identical fair dice is P_m. If the dice have 6 (six) faces labeled by $1, 2, \ldots 6$, which of the following statements is correct ?

(a) $P_9 = P_5$ (b) $P_9 = P_4$ (c) $P_9 = P_3$ (d) $P_9 = P_6$

Ans. 2: (a)

Solution.:

P_9	First Dice	Second Dice	Sum	
Total events = 36	3	6	9	$P_9 = \frac{4}{36} = \frac{1}{9}$
	4	5	9	
	5	4	9	
	6	3	9	
P_5	1	4	5	$P_5 = \frac{4}{36} = \frac{1}{9}$
	2	3	5	
	3	2	5	
	4	1	5	
P_4	1	3	4	$P_4 = \frac{3}{36} = \frac{1}{12}$
	2	2	4	
	3	1	4	
P_6	1	5	6	$P_6 = \frac{5}{36}$
	2	4	6	
	3	3	6	
	4	2	6	
	5	1	6	

Clearly $P_5 = P_9$. Thus 'a' is correct.

Q12. If θ and ϕ are respectively the polar and azimuthal angles on the unit sphere, what is $\left\langle \cos^2(\theta) \right\rangle$ and $\left\langle \sin^2(\theta) \right\rangle$, where $\langle O \rangle$ denotes the average of O?

(a) $\left\langle \cos^2(\theta) \right\rangle = 2/3$ and $\left\langle \sin^2(\theta) \right\rangle = 1/3$

(b) $\left\langle \cos^2(\theta) \right\rangle = 1/2$ and $\left\langle \sin^2(\theta) \right\rangle = 1/2$

(c) $\left\langle \cos^2(\theta) \right\rangle = 3/4$ and $\left\langle \sin^2(\theta) \right\rangle = 1/4$

(d) $\left\langle \cos^2(\theta) \right\rangle = 1/3$ and $\left\langle \sin^2(\theta) \right\rangle = 2/3$

Ans. 12: (d)

Solution.:

At the surface of unit sphere

$da = \sin\theta d\theta d\phi$ (as $R = 1$)

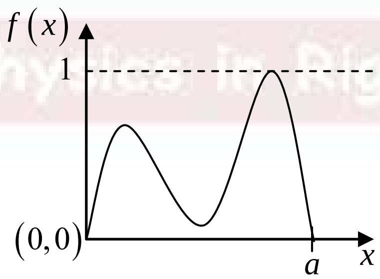
$$\left\langle \cos^2\theta \right\rangle = \frac{\iint \cos^2\theta \sin\theta d\theta d\phi}{\iint \sin\theta d\theta d\phi} = \frac{\int_0^{\pi} \cos^2\theta \sin\theta d\theta \int_0^{2\pi} d\phi}{4\pi} = -\left|\frac{\cos^3\theta}{3}\right|_0^{\pi} \cdot \frac{2\cancel{\pi}}{4\cancel{\pi}}\frac{1}{2} = -\frac{2}{3} \times \frac{1}{2} = \frac{1}{3}$$

$$\left\langle \sin^2\theta \right\rangle = \frac{\int_0^{\pi} \sin^2\theta \sin\theta d\theta \int_0^{2\pi} d\phi}{\iint \sin\theta d\theta d\phi} = \frac{2\pi}{4\pi}\int_0^{\pi}\left(1-\cos^2\theta\right)\sin\theta d\theta = \frac{1}{2}\left[\left|\cos\theta\right|_0^{\pi} - \int_0^{\pi}\cos^2\theta \sin\theta d\theta\right]$$

$$= \frac{1}{2}\left[2 + \left|\frac{\cos^3\theta}{3}\right|_0^{\pi}\right] = \frac{1}{2}\left[2 - \frac{2}{3}\right] = \frac{1}{2}\cdot\frac{4}{3} = \frac{2}{3}$$

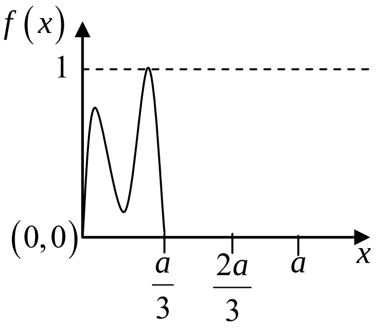
Thus, (a) is correct.

Q13. The function $f(x)$ shown below has non-zero values only in the range $0 < x < a$.

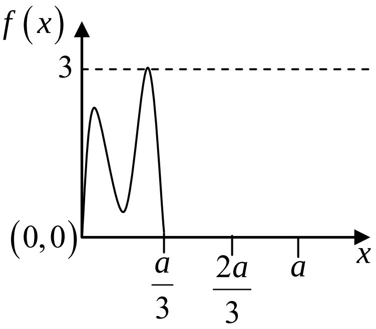


Which of the following figure represents $f(3x)$?

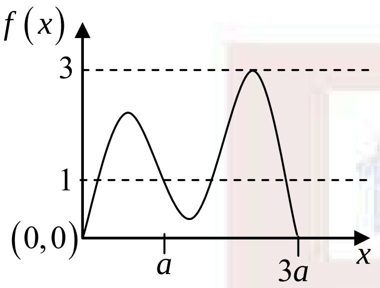
(a)



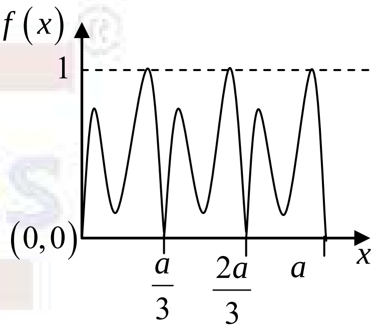
(b)



(c)



(d)



Ans. 13: (a)

Solution.:

If $f(x)$ has period "a" then $f(3x)$ will have period $\frac{a}{3}$.

Maximum value of the function $f(x)$ and $f(3x)$ will be same.

As $f(x)$ is zero beyond $x = a$, so 'd' is ruled out.

As for $f[3x]$, $3\cdot\frac{a}{3}$ a [which happens for x beyond $\frac{a}{3}$]

Thus 'a' is correct option.

Q14. Consider a complex function

$$f(z) = \frac{1}{6z^3 + 3z^2 + 2z + 1}$$

What is the sum of the residues at its poles?

(a) $\frac{i\sqrt{3}}{7}$ (b) $\frac{4}{7}$ (c) $\frac{2}{7}$ (d) 0

Ans. 14: (d)

Solution. :

$$f(z)=\frac{1}{6z^3+3z^2+2z+1}=\frac{1}{3z^2(2z+1)+1(2z+1)}=\frac{1}{(3z^2+1)(2z+1)}$$

Thus poles are $z=-\frac{1}{2}$ and $z=\pm\frac{i}{\sqrt{3}}$

$$\text{Res}\left(z=-\frac{1}{2}\right)=\frac{1}{\left(6z^3+3z^2+2z+1\right)'\Big|_{z=-\frac{1}{2}}}=\frac{1}{\left|18z^2+6z+2\right|_{z=-\frac{1}{2}}}$$

$$=\frac{1}{18\cdot\frac{1}{4}-6\cdot\frac{1}{2}+2}=\frac{1}{\frac{9}{2}-3+2}=\frac{1}{\frac{9}{2}-1}=\frac{2}{7} \quad ....(1)$$

$$\text{Res}\left(z=\frac{i}{\sqrt{3}}\right)=\frac{1}{\left(6z^3+3z^2+2z\right)_{z=\frac{i}{\sqrt{3}}}}=\frac{1}{\left|18z^2+6z+2\right|_{z=\frac{i}{\sqrt{3}}}}=\frac{1}{-\frac{18}{3}+\frac{6i}{\sqrt{3}}+2}$$

$$=\frac{1}{-6+2\sqrt{3}i+2}=\frac{1}{-4+2\sqrt{3}i}\times\frac{-4-2\sqrt{3}i}{-4-2\sqrt{3}i}=\frac{-4-2\sqrt{3}i}{16+12}=\frac{-2\sqrt{3}i-4}{28} \quad ....(2)$$

$$\text{Res}\left(z=-\frac{i}{\sqrt{3}}\right)=\frac{1}{\left(18z^2+6z+2\right)_{z=-\frac{i}{\sqrt{3}}}}=\frac{1}{\left|18z^2+6z+2\right|_{z=-\frac{i}{\sqrt{3}}}}=\frac{1}{-\frac{18}{3}-\frac{6i}{\sqrt{3}}+2}$$

$$=\frac{1}{-6-2\sqrt{3}i+2}=\frac{1}{-4-2\sqrt{3}i}\times\frac{-4+2\sqrt{3}i}{-4+2\sqrt{3}i}=\frac{-4+2\sqrt{3}i}{16+12}=\frac{2\sqrt{3}i-4}{28} \quad ....(3)$$

$$\text{Sum of Residue}=\frac{2}{7}+\frac{-2\sqrt{3}i-4}{28}+\frac{2\sqrt{3}i-4}{28}=\frac{2}{7}-\frac{1}{7}-\frac{1}{7}=0$$

Q15. Consider a complex number $z=x+iy$. Where do all the zeros of $\cos(z)$ lie?

(a) On the $x=y$ line.
(b) On the $x=0$ line.
(c) On the $y=0$ line.
(d) On the $x=-y$ line.

Ans. 15: (c)

Solution.: $\cos z=0=\cos\left((2n+1)\frac{\pi}{2}\right)\Rightarrow z=(2n+1)\frac{\pi}{2}\Rightarrow x+iy=(2n+1)\frac{\pi}{2}\Rightarrow y=0$

Hence zero's of $\cos z$ will lie on x-axis i.e. $y=0$ line.

Thus, 'c' is correct.

Part-B: 3-Mark Questions

Q10. $G=\left(e,a,a^2,b,ba,ba^2\right)$ is a group of order 6. e is the identity element and a is of order 3. What could be the order of the element b?

(a) 3 (b) 2 (c) 1 (d) Can't be determined

Ans. 10: (b)

Solution. : $G=\left\{e,a,a^2,b,ba,ba^2\right\}$ is of order 6. Let order of $b=p$

Consider group multiplication table

	e	a	a^2	b	ba	ba^2
e						
a						
a^2						
b	b	ba	ba^2	b^2	b^2a	b^2a^2
ba						
ba^2						

In the highlighted row, all the elements of the group should be reproduced. b, ba and ba^2 are already there. Thus

$b^2 \rightarrow e$

$b^2a \rightarrow$ should be equal to a

$b^2a^2 \rightarrow a^2$

If $b^2=e$, $b^2a=a$, $b^2a^2=a^2$

All six elements are reproduced. Order of b must be 2. Thus 'b' is correct.

Part-C: 2-Mark Numerical Questions

Q7. Let $M=2\mathrm{I}+\sigma_x+i\sigma_y+\sigma_z$ is a 2×2 square matrix, where, σ_α denotes α^{th} Pauli matrix, and I denotes the 2×2 identity matrix. It is given that $|u\rangle=\begin{pmatrix}1\\0\end{pmatrix}$ and $|v\rangle=\begin{pmatrix}1\\-1\end{pmatrix}$ are column vectors. What is the value of $\left\langle u\left|\sqrt{M}\right|v\right\rangle$?

Ans.: 1.73

Solution. :

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ \sigma_y = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \text{ and } \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$M = 2\mathrm{I} + \sigma_x + i\sigma_y + \sigma_z = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 2 & 1 \end{pmatrix}$$

$$\Rightarrow M = \begin{pmatrix} 3 & 0 \\ 2 & 1 \end{pmatrix}$$

Eigen values of matrix $M = \begin{pmatrix} 3 & 0 \\ 2 & 1 \end{pmatrix}$ are $\lambda = 1$ and 3.

We can write $f(M) = \alpha_0 I + \alpha_1 M = \sqrt{M}$. Thus $\alpha_0 I + \alpha_1 \lambda = \sqrt{\lambda}$.

$\alpha_0 + \alpha_1 = 1$ and $\alpha_0 + 3\alpha_1 = \sqrt{3}$. Solving this we get $\alpha_0 = \dfrac{3-\sqrt{3}}{2}$ and $\alpha_1 = \dfrac{\sqrt{3}-1}{2}$.

$$\text{Now } \sqrt{M} = \alpha_0 I + \alpha_1 M = \begin{pmatrix} \frac{3-\sqrt{3}}{2} & 0 \\ 0 & \frac{3-\sqrt{3}}{2} \end{pmatrix} + \begin{pmatrix} \frac{3\sqrt{3}-3}{2} & 0 \\ \frac{2\sqrt{3}-2}{2} & \frac{\sqrt{3}-1}{2} \end{pmatrix}$$

$$\Rightarrow \sqrt{M} = \begin{pmatrix} \frac{3-\sqrt{3}}{2} & 0 \\ 0 & \frac{3-\sqrt{3}}{2} \end{pmatrix} + \begin{pmatrix} \frac{3\sqrt{3}-3}{2} & 0 \\ \frac{2\sqrt{3}-2}{2} & \frac{\sqrt{3}-1}{2} \end{pmatrix} = \begin{pmatrix} \sqrt{3} & 0 \\ \sqrt{3}-1 & 1 \end{pmatrix}$$

$$\left\langle u \middle| \sqrt{M} \middle| \upsilon \right\rangle = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} \sqrt{3} & 0 \\ \sqrt{3}-1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} \sqrt{3} \\ \sqrt{3}-2 \end{pmatrix} = \left(\sqrt{3}\right) = 1.73$$

Part-A: 1-Mark Questions

Q1. For a system of unit mass, the dynamical variables follow the relation $\dot{x}^2 = kx_0^2 + \dot{x}_0^2 - kx^2$ where, x is the position of the system at time t, and x_0 is its initial position. What is the force acting on the system?

(a) $-k(x-x_0)$ (b) $-kx$ (c) $-\frac{1}{2}k(x-x_0)$ (d) $\frac{1}{2}k(x-x_0)^2$

Ans. 1: (b)

Solution: Generalised force $Q = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}}\right) - \frac{\partial T}{\partial x}$

$$\because T = \frac{1}{2}\dot{x}^2 = \frac{1}{2}\left(kx_0^2 + \dot{x}_0^2 - kx^2\right) \Rightarrow Q = kx$$

Q3. A particle of mass m is moving in a circular path of constant radius r such that its centripetal acceleration a_c is varying with time t as $a_c = k^2rt^2$ where, k is a constant. The power delivered to the particle by the force acting on it is

(a) $\frac{1}{2}mk^2r^2t$ (b) $2\pi mk^{\frac{3}{2}}r^2$ (c) mk^2r^2t (d) 0

Ans. 3: (c)

Solution: For circular motion: $\frac{mv_\theta^2}{r} = mk^2rt^2$

$$k = \frac{1}{2}mv_\theta^2 = \frac{1}{2}mk^2r^2t^2;\; P = \frac{dW}{dt} = \frac{dk}{dt} = mk^2r^2t$$

Q4. The front-end of a train moving with constant acceleration, passes a pole with velocity u, and its back-end passes the pole with velocity v. With what velocity does the mid-point of this train pass the same pole?

(a) $\frac{1}{2}\sqrt{u^2+v^2}$ (b) $\sqrt{\frac{u^2+v^2}{2}}$ (c) $\frac{uv}{u+v}$ (d) $\frac{u+v}{2}$

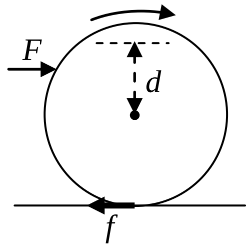
Ans. 4: (b)

Solution:

$$a = \frac{v^2-u^2}{2L} = \frac{V^2-u^2}{2(L/2)} \qquad \Rightarrow 2V^2 = v^2+u^2 \qquad \Rightarrow V = \sqrt{\frac{v^2+u^2}{2}}$$

Part-B: 3-Mark Questions

Q1. A cylinder of radius R is constrained to roll without slipping on a horizontal plane under the action of a constant force F applied d distance above the axis of the cylinder. In the process, it experiences a frictional force f at the point of contact (see figure). For what value of d, the magnitude of f is minimum?



(a) $-R/2$

(b) R

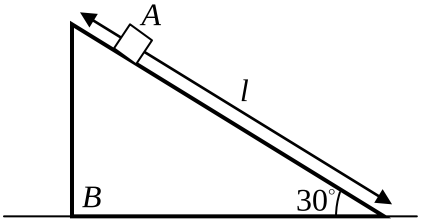
(c) $R/2$

(d) $-R$

Ans. 1: (c)

Solution: f will be zero when $d=\dfrac{I_{CM}}{MR}=\dfrac{\frac{1}{2}MR^2}{MR}=\dfrac{R}{2}$

Q2. A small object A of mass m is free to slide on the inclined plane of a triangular block B of mass $2m$ (see figure). Initially both the blocks are motionless. Block A starts sliding under the action of gravity from the highest point of block B. What is the speed of block B, when block A hits the floor?

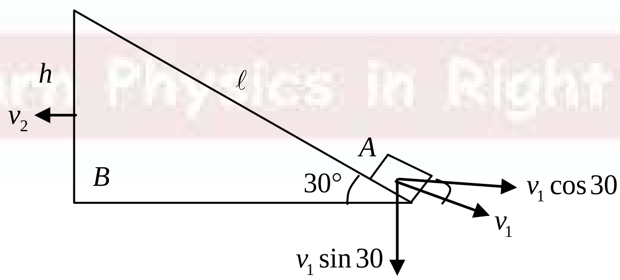


(a) $\frac{1}{2}\sqrt{gl}$ (b) $\frac{1}{3}\sqrt{gl}$ (b) $\sqrt{gl}$ (d) $\frac{2}{3}\sqrt{gl}$

Ans. 2: (b)

Solution: Let v_1 is the velocity of block A with respect to block B, when it hits the floor.

Let v_2 is the velocity of block B at the same time.



$v_{AE}=v_{AB}-v_{EB}$

$v_X=v_1\cos 30°-v_2$; $v_Y=v_1\sin 30°$ $\quad\quad$ $\left(\text{Here } v_{BE}=-v_2\right)$

v_X and v_Y are the velocities with respect to earth.

Conservation of linear momentum; $m\left(v_1\cos 30°-v_2\right)-2mv_2=0\Rightarrow v_1=2\sqrt{3}v_2$

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Conservation of energy; $mgh = \frac{1}{2}mv_X^2 + \frac{1}{2}mv_Y^2 + \frac{1}{2}(2m)v_Z^2$

$$\Rightarrow mgh = \frac{1}{2}m\left(v_1\cos 30^\circ - v_2\right)^2 + \frac{1}{2}m\left(v_1\sin 30^\circ\right) + mv_2^2$$

$$\Rightarrow gh = 2v_2^2 + \frac{3}{2}v_2^2 + v_2^2 = \frac{9}{2}v_2^2 \Rightarrow v_2 = \sqrt{\frac{2gh}{9}} = \sqrt{\frac{2g}{9}\frac{\ell}{2}} = \frac{1}{3}\sqrt{g\ell}$$

Q3. A particle moving in a central force field centered at $r=0$, follows a trajectory given by $r = e^{-\alpha\theta}$ where, (r,θ) is the polar coordinate of the particle and $\alpha > 0$ is a constant. The magnitude of the force is proportional to

(a) r^{-3} (b) r^2 (c) r^{-1} (d) r^3

Ans. 3: (a)

Solution: $r = e^{-\alpha\theta} \quad \Rightarrow u = e^{\alpha\theta} \quad \left(\because u = \frac{1}{r}\right) \quad \Rightarrow \frac{d^2u}{d\theta^2} = \alpha^2 e^{\alpha\theta}$

Differential equation of the orbit; $\frac{d^2u}{d\theta^2} + u = -\frac{m}{\ell^2u^2} f\left(\frac{1}{u}\right)$

$$\alpha^2 e^{\alpha\theta} + e^{\alpha\theta} = -\frac{m}{\ell^2 e^{2\theta}} f\left(\frac{1}{u}\right) \Rightarrow f\left(\frac{1}{u}\right) = -\frac{\ell^2}{m}\left(1+\alpha^2\right)e^{3\theta} \Rightarrow f(r) = -\frac{\ell^2}{mr^3}\left(1+\alpha^2\right)$$

Q6. The Lagrangian of a particle of unit mass is given by $L = \frac{1}{2}\left(\dot{x}^2 - x^2 + 2x\dot{x}\right)$. The Hamiltonian of this system is given by

(a) $\frac{1}{2}p^2 - px + x^2$ (b) $\frac{1}{2}\left(p^2 + x^2\right)$ (c) $\frac{1}{2}(p-x)^2$ (d) $\frac{1}{2}p^2 + px - x^2$

Ans. 6: (a)

Solution:

$$L = \frac{1}{2}\left(\dot{x}^2 - x^2 + 2x\dot{x}\right); \quad p = \frac{\partial L}{\partial \dot{x}} = \dot{x} + x \quad \Rightarrow \dot{x} = p - x$$

$$H = p\dot{x} - L = p\dot{x} - \frac{1}{2}\dot{x}^2 + \frac{1}{2}x^2 - x\dot{x} = p(p-x) - \frac{1}{2}(p-x)^2 + \frac{1}{2}x^2 - x(p-x)$$

$$\Rightarrow H = p^2 - px - \frac{1}{2}p^2 - \frac{1}{2}x^2 + px + \frac{1}{2}x^2 - px + x^2 \quad \Rightarrow H = \frac{1}{2}p^2 - px + x^2$$

Q12. If three real variables x, y and z evolve with time t following

$$\frac{dx}{dt} = x(y-z), \frac{dy}{dt} = y(z-x), \frac{dz}{dt} = z(x-y),$$

then which of the following quantities remains invariant in time ?

(a) $xy+yz+zx$ (b) $x^2+y^2+z^2$ (c) $\frac{1}{xy}+\frac{1}{yz}+\frac{1}{zx}$ (d) $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}$

Ans. 12: (c)

Solution:

$$\frac{d}{dt}\left(\frac{1}{xy}+\frac{1}{yz}+\frac{1}{zx}\right)=\frac{x\,dy/dt+y\,dx/dt}{x^2y^2}+\frac{y\dot{z}+z\dot{y}}{y^2z^2}+\frac{z\dot{x}+x\dot{z}}{x^2z^2}$$

$$=\frac{xz^2\dot{y}+yz^2\dot{x}+yx^2\dot{z}+zx^2\dot{y}+zy^2\dot{x}+xy^2\dot{z}}{x^2y^2z^2}$$

$$=\frac{yz(z+y)\dot{x}+zx(z+x)\dot{y}+xy(x+y)\dot{z}}{x^2y^2z^2}$$

$\dot{x}=x(y-z),\ \dot{y}=y(z-x),\ \dot{z}=z(x-y)$

$$\frac{d}{dt}\left(\frac{1}{xy}+\frac{1}{yz}+\frac{1}{zx}\right)=\frac{xyz\left(y^2-z^2\right)+xyz\left(z^2-x^2\right)+xyz\left(x^2-y^2\right)}{x^2y^2z^2}=0$$

So, $\frac{1}{xy}+\frac{1}{yz}+\frac{1}{zx}$ = constant

Q13. A circularly polarized laser of power P is incident on a particle of mass m. The particle, which was initially at rest, completely absorbs the incident radiation. The kinetic energy of the particle as a function of time t is given by

(a) $\frac{1}{2}Pt\left(\frac{Pt}{mc^2}+1\right)$ (b) $\frac{1}{2}Pt\left(\frac{Pt}{mc^2}-1\right)$ (c) $\frac{P^2t^2}{2mc^2}$ (d) $\frac{Pt}{2}$

Ans. 13: (a)

Solution:

In case of circulary polarized light, the energy absorbed by the particle and its kinetic energy has following relation:

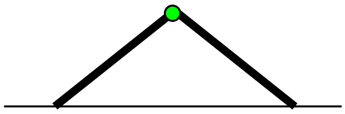
$$E=\sqrt{mc^2\left(2T-E\right)}$$

$$pt=\sqrt{mc^2\left(2T-pt\right)}\Rightarrow\frac{p^2t^2}{mc^2}=2T-pt\Rightarrow 2T=pt\left(\frac{pt}{mc^2}+1\right)$$

$$\Rightarrow T=\frac{1}{2}pt\left(\frac{pt}{mc^2}+1\right)$$

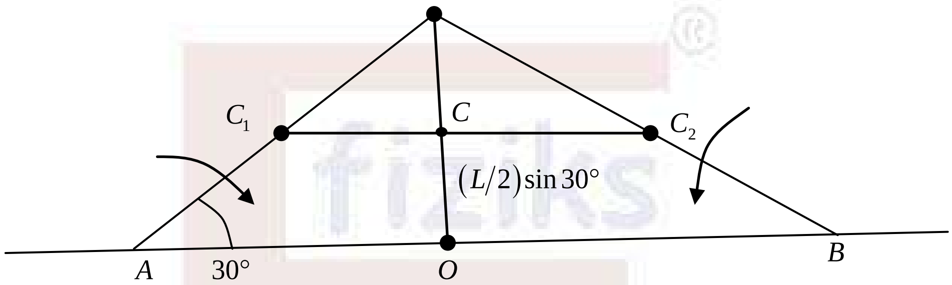
Part-C: 2-Mark Numerical Questions

Q1. Two uniform rods of length $1m$ are connected to a friction-less hinge A. The hinge is held at a height and the other ends of the rods rests on a friction-less plane, such that the angle between the rods is $2\pi/3$. If the hinge is released from the rest, what is the speed of the hinge when it hits the floor? [Acceleration due to gravity is $9.81ms^{-2}$]



Ans.: 1.92

Solution:



In this process, the centre of mass will fall in the downward direction by a distance $\frac{L}{2}\sin 30°$.

$$E_i = (2M)g\left(\frac{L}{2}\sin 30°\right) = \frac{1}{2}MgL$$

when C will hit the ground, the system will have translational as well as rotational kinetic energy

$$E_f = \frac{1}{2}(2M)v^2 + \frac{1}{2}I_A\omega^2 + \frac{1}{2}I_B\omega^2 = mv^2 + 2\cdot\frac{1}{2}\cdot\frac{ML^2}{3}\cdot\frac{v^2}{L^2} = \frac{4}{3}Mv^2$$

$$\because E_f = E_i \quad \Rightarrow \frac{4}{3}Mv^2 = \frac{1}{2}MgL \Rightarrow v = \sqrt{\frac{3gL}{8}} = \sqrt{\frac{3\times 9.8\times 1}{8}} = 1.92$$

Part-A: 1-Mark Questions

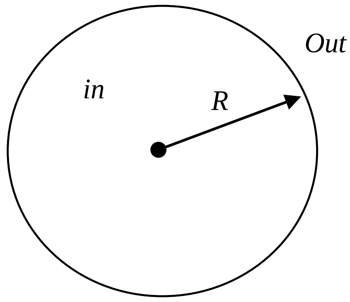
Q9. A conducting sphere of radius R is placed in a uniform electric field E_0 directed along $+z$ axis. The electric potential for outside points is given by $V_{out} = -E_0\left(1-\left(R/r\right)^3\right)r\cos\theta$, where r is the distance from the center and θ is the polar angle. The charge density on the surface of the sphere is

(a) $3\in_0 E_0\cos\theta$ (b) $\in_0 E_0\cos\theta$ (c) $-3\in_0 E_0\cos\theta$ (d) $\frac{1}{3}\in_0 E_0\cos\theta$

Ans. 9: (a)

Solution. :

We know perpendicular component of electric field is discontinuous across a surface carrying free charge



$E_{out}^{\perp} - E_{in}^{\perp} = \frac{\sigma}{\varepsilon_0}$ at $r = R$ ....(1)

Inside conducting sphere $E_{in}^{\perp} = 0$

$$E_{out}^{\perp} = -\frac{\partial V_{out}}{\partial r}$$

$$E_{out}^{\perp}\Big|_{r=R} = -\frac{\partial}{\partial r}\left|-E_0 r\cos\theta + E_0\frac{R^3}{r^2}\cos\theta\right|_{r=R} = -\left[E_0\cos\theta - 2E_0\frac{R^3}{r^3}\cos\theta\right]_{r=R}$$

$$= -\left[-E_0\cos\theta - 2E_0\frac{\cancel{R^3}}{\cancel{R^3}}\cos\theta\right] = 3E_0\cos\theta \rightarrow \text{Put in (1) we get}$$

$$\frac{\sigma}{\varepsilon_0} = 3E_0\cos\theta \Rightarrow \sigma = 3\varepsilon_0 E_0\cos\theta$$

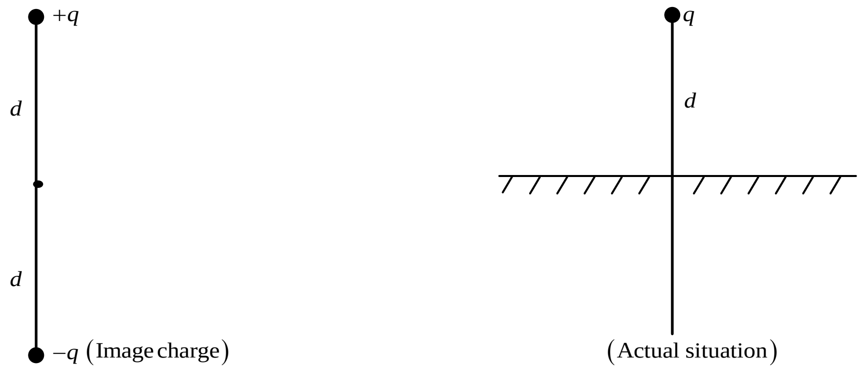
Thus, (a) is correct.

Q10. A point charge q is kept d distance above an infinite conducting plane. What is the energy stored in the configuration?

(a) $-\frac{1}{4\pi\in_0}\frac{q^2}{2d}$ (b) $-\frac{1}{4\pi\in_0}\frac{q^2}{4d}$ (c) $\frac{1}{4\pi\in_0}\frac{q^2}{2d}$ (d) $\frac{1}{4\pi\in_0}\frac{q^2}{4d}$

Ans. 10: (b)

Solution.:

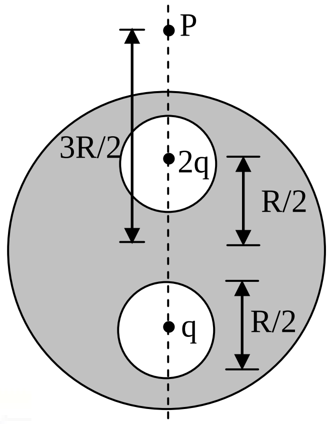


Potential energy of interaction between $= \frac{1}{2}$ a charge (separated by distance $2d$)

Potential energy of interaction between two equal and opposite charges separated by $2d$

$$= \frac{1}{2}\left[\frac{-q \times q}{4\pi\varepsilon_0 \cdot 2d}\right] = \frac{-q^2}{4\pi\varepsilon_0 \cdot 4d}.$$ Thus (b) is correct.

Q11. Two point charges $2q$ and q are placed inside two spherical cavities of equal radii $R/4$ in a solid conducting sphere of radius R, as shown in the figure. The cavities are placed along a diagonal at distances $R/2$ from the center of the solid sphere. The electrical potential at a point $P, 3R/2$ distance away from the center along the same diagonal, is given by



(a) 0 (b) $\frac{1}{4\pi \in_0} \frac{5q}{2R}$

(c) $\frac{1}{4\pi \in_0} \frac{2q}{R}$ (d) $\frac{1}{4\pi \in_0} \frac{3q}{R}$

Ans. 11: (c)

Solution.:

The $2q$ charge will induce $-2q$ charge at the surface of the cavity which will induce $2q$ at the surface of sphere.

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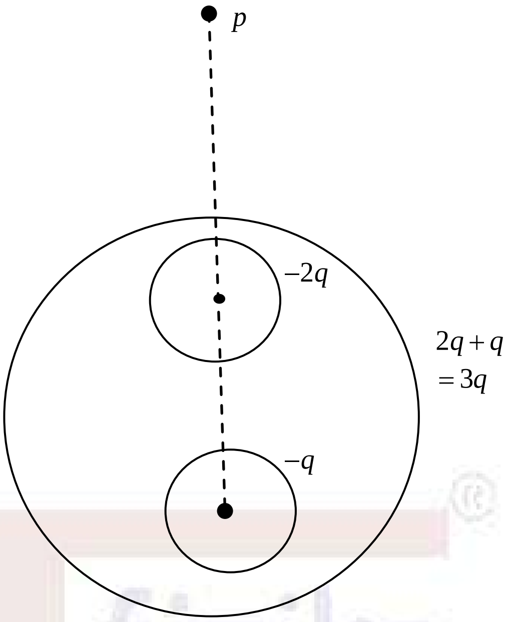


Similarly, 'q' in second cavity will induce 'q' at the surface of the sphere.

Total charge on surface of sphere $=3q$.

In spherical symmetry on the distance of point under observation from the centre of charge distribution matters.

Thus, $V_{CP}=\frac{3R}{2}=\frac{1}{4\pi\varepsilon_0}\times\frac{\cancel{3}q}{\cancel{3}R/2}=\frac{2q}{4\pi\varepsilon_0 R}$

Hence (c) is correct option.

Part-B: 3-Mark Questions

Q8. A point charge q is fixed at point A inside a hollow grounded conducting spherical shell of radius R, at a distance a from the center C. The force on the sphere due to the presence of the point charge is

(a) $\frac{1}{4\pi\in_0}\frac{q^2aR}{(R+a)^2(R-a)^2}$ in magnitude and along $\overrightarrow{AC}$.

(b) $\frac{1}{4\pi\in_0}\frac{q^2aR}{(R+a)^2(R-a)^2}$ in magnitude and along $\overrightarrow{CA}$.

(c) $\frac{1}{4\pi\in_0}\frac{q^2}{(R-a)^2}$ in magnitude and along $\overrightarrow{AC}$.

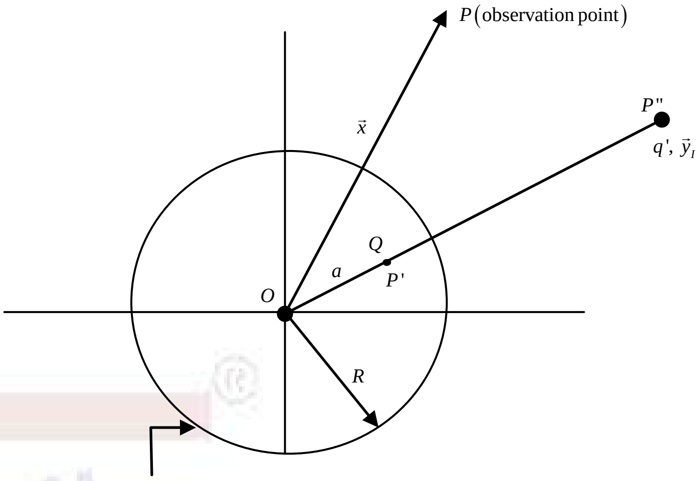
(d) $\frac{1}{4\pi\in_0}\frac{q^2}{(R-a)^2}$ in magnitude and along $\overrightarrow{CA}$.

Ans. 8: (a)

Solution.:

Source charge is at 'a' of magnitude 'Q'

Let unit vector along the line joining centre and location of 'Q' is $\hat{n}_1$



$\Rightarrow \overrightarrow{OP'} = a\hat{n}_1$

In the corresponding image problem, image charge will be along same direction at $P''(\vec{y})$.

$\overrightarrow{OP''} = \vec{y}_I = y_I\hat{n}_1$

Net potential at $\vec{x}$ is

$$\Phi(\vec{x}) = \frac{Q}{4\pi\varepsilon_0 |x\hat{n} - a\hat{n}_1|} + \frac{q'}{4\pi\varepsilon_0 |x\hat{n} - y_I\hat{n}_1|}$$

Now our boundary condition (original) is that $\Phi(x = R) = 0$, as sphere was grounded.

$$\Phi(\vec{R}) = \frac{Q}{4\pi\varepsilon_0 |R\hat{n} - a\hat{n}_1|} + \frac{q'}{4\pi\varepsilon_0 |R\hat{n} - y_I\hat{n}_1|} = 0$$

$$\Rightarrow \Phi(\vec{R}) = \frac{Q}{4\pi\varepsilon_0 R\left|\hat{n} - \frac{a}{R}\hat{n}_1\right|} + \frac{q'}{4\pi\varepsilon_0 y_I\left|\frac{R}{y_I}\hat{n} - \hat{n}_1\right|} = 0$$

Equality will be satisfied, if

$$\frac{Q}{4\pi\varepsilon_0 R} = -\frac{q'}{4\pi\varepsilon_0 y_I} \Rightarrow q' = \frac{-Q}{R}y_I \qquad \text{and } \frac{a}{R} = \frac{R}{y_I} \Rightarrow y_I = \frac{R^2}{a}$$

Force of attraction between 'Q' and grounded sphere = Force between 'Q' and its image

$$\vec{F}_{Qq'} = \frac{1}{4\pi\varepsilon_0}\frac{Qq'}{|y_I - a|^2}\hat{n}_1 = \frac{1}{4\pi\varepsilon_0}\frac{Q\times\left(-\frac{Q}{\cancel{R}}\cdot\frac{R^{\cancel{2}}}{a}\right)}{\left|\frac{R^2}{a} - a\right|^2}\hat{n}_1 = -\frac{1}{4\pi\varepsilon_0}\frac{Q^2}{|R^2 - a^2|^2}\cdot R\frac{a^{\cancel{2}}}{\cancel{a}}\hat{n}_1$$

$$= \frac{1}{4\pi\varepsilon_0}\frac{Q^2 Ra}{(R-a)^2(R+a)^2}(-\hat{n}_1). \qquad -\hat{n}_1 \text{ is along } \overrightarrow{AC}$$

Hence (a) is correct

Q9. A rectangular dielectric slab partly fills two identical rectangular parallel plate capacitors which are maintained at potentials V_1 and V_2 with $V_1 > V_2$. The slab can freely move in the space between the capacitor plates without any friction. Which of the following is true?

(a) The slab will not move.

(b) The slab will move towards lower potential.

(c) The slab will move towards higher potential.

(d) The slab will position itself at $1/V_1 : 1/V_2$ ratio between capacitors 1 and 2.

Ans. 9: (c)

Solution 5:

Force on dielectric $= \frac{1}{2}V^2\frac{dC}{dx}$. Thus $F \propto V^2$, $\frac{dc}{dx}$ has a small effect.

As $V_1 > V_2$

$\Rightarrow$ Force towards higher potential will be more. This, slab will move towards higher potential.

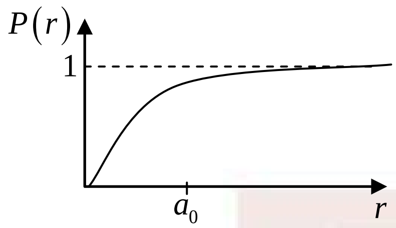
Part-C: 2-Mark Numerical Questions

No Question

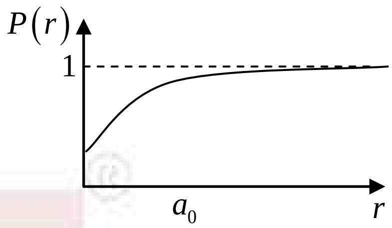
Part-A: 1-Mark Questions

Q17. The wave function of the electron in a Hydrogen atom in a particular state is given by $\pi^{-1/2}a_0^{-3/2}\exp\left(-r/a_0\right)$. Which of the following figures qualitatively depicts the probability $\left(P(r)\right)$ of the electron to be within a distance r from the nucleus?

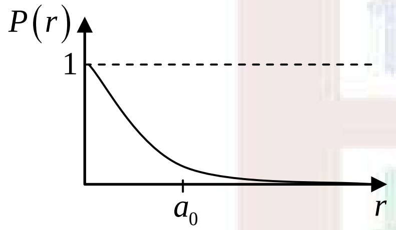
(a)



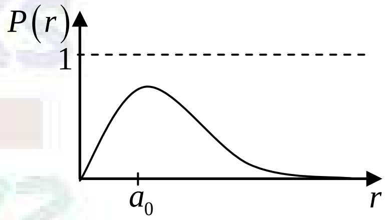
(b)



(c)



(d)



Ans. 17: (a)

17. Ans. (d)

Solution:

$$P(r)=r^2\left|\psi(r)\right|^2=\frac{1}{\pi a^3}r^2e^{-2r/a_0} \text{ and } \frac{dP(r)}{dr}=0 \Rightarrow 2re^{-2r/a_0}-\frac{2}{a_0}r^2e^{-2r/a_0}=0 \Rightarrow \boxed{r=a_0}$$

Q22. A beam of high energy neutrons is scattered from a metal lattice, where the spacing between nuclei is around $0.4\,nm$. In order to see quantum diffraction effects, the kinetic energy of the neutrons must be of the order [Mass of neutron $=1.67\times10^{-27}\,kg$, Planck's constant $=6.62\times10^{-34}\,m^2kgs^{-1}$]

(a) eV (b) MeV (c) meV (d) keV

Ans. 22: (c)

Solution:

$$2d\sin\theta=\lambda \Rightarrow \lambda=2d$$

Since $d=0.4\,\text{nm} \Rightarrow \lambda=1.6\,\text{nm}$ where $\lambda=\frac{0.28}{\sqrt{E(eV)}}\,\text{Å}$

$$\Rightarrow E(eV)=\frac{(0.2r)^2}{\left[\lambda(\text{Å})\right]^2}=\frac{0.0784}{(16)^2}=\frac{0.0784}{256} \Rightarrow E=3\times10^{-4}\,\text{eV}=30\,\text{meV} \Rightarrow \boxed{E\approx\text{meV}}$$

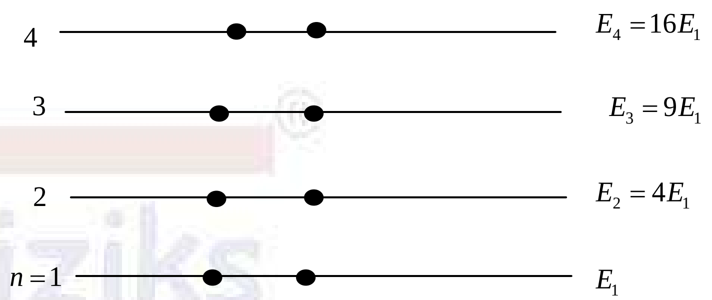
Q23. Consider eight electrons confined in a $1D$ box of length d. What is the minimum total energy for the system allowed by Pauli's exclusion principle?

(a) $\frac{15h^2}{4md^2}$ (b) $\frac{15h^2}{2md^2}$ (c) $\frac{30h^2}{md^2}$ (d) $\frac{15h^2}{8md^2}$

Ans. 23: (b)

23. Ans.(c)

Solution:



$$E_1 = \frac{\pi^2\hbar^2}{2md^2}$$

$$\therefore\ E = 2E_1 + 2E_2 + 2E_3 + 2E_4$$

$$= 2(E_1 + E_2 + E_3 + E_4)$$

$$= 2(E_1 + 4E_1 + 9E_1 + 16E_1)$$

$$= 60E_1 = 60 \times \frac{\pi^2\hbar^2}{2md^2} = \frac{30\pi^2\hbar^2}{md^2}$$

Q24. Consider 5 identical spin $\frac{1}{2}$ particles moving in a 3-dimensional harmonic oscillator potential,

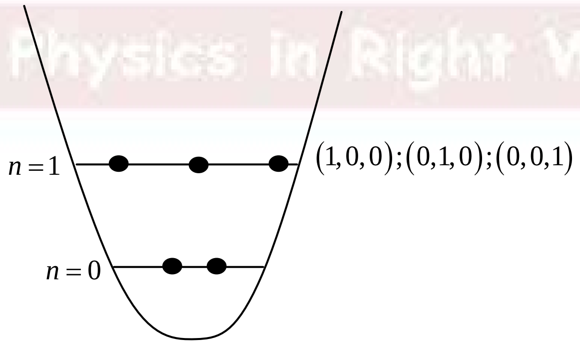
$$V(r) = \frac{1}{2}m\omega^2 r^2 = \frac{1}{2}m\omega^2\left(x^2 + y^2 + z^2\right)$$

The degeneracy of the ground state of the system is

(a) 5 (b) 7 (c) 20 (d) 32

Ans. 24: (c)

Solution:



Two spin-$\frac{1}{2}$ particles are in the ground state, $n = 0$.

Three particles are arranged in $n = 1$ (degeneracy = 6 including spin).

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Ground state energy $E = 2\times\frac{3}{2}\hbar\omega + 3\times\frac{5}{2}\hbar\omega = \frac{21}{2}\hbar\omega$

Total possible combinations of arranging 3 spin-$\frac{1}{2}$ particles in $n=1$ state are

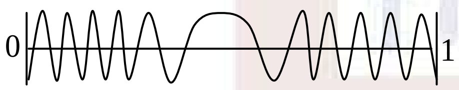
$$^6C_3 = \frac{6!}{3!3!} = 20$$

Q25. A particle is confined in an infinite potential well of the form given below.

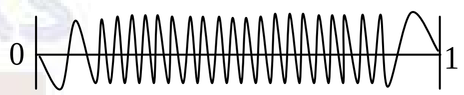
$$V(x) = \begin{cases} 4V_0 x(1-x), & \forall 0 \le x \le 1 \\ \infty & \text{otherwise} \end{cases}$$

If the particle has energy $E \ge V_0$, which of the following could be the form of its wave function?

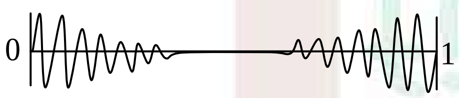
(a)



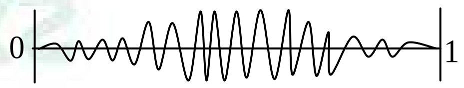
(b)



(c)



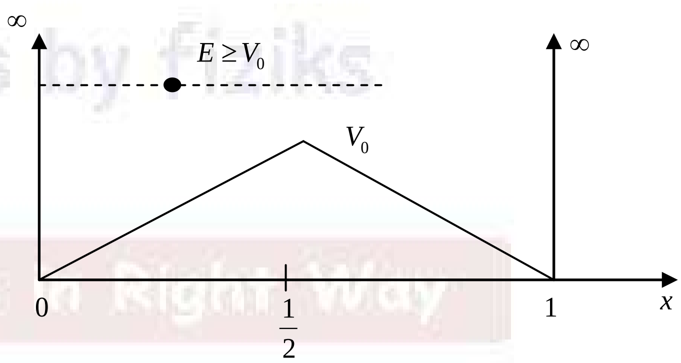
(d)



Ans. 25: (a)

Solution:

$$V(x) = \begin{cases} 4V_0 x(1-x): & 0 \le x \le 1 \\ \infty & : \text{ otherwise} \end{cases}$$



Wavelength $\lambda = \frac{\hbar}{p}$ where

$p = \sqrt{2m(E - V_0)}$

Thus λ will increase as p decreases from $x=0$ to $x = 1/2$ and then λ decreases as p increases from $x = \frac{1}{2}$ to $x = 1$.

Therefore correct option is (a).

Part-B: 3-Mark Questions

Q4. For a one dimensional simple harmonic oscillator, for which $|0\rangle$ denotes the ground state, what is the constant β in

$$\langle 0|e^{ikx}|0\rangle = e^{-\beta\langle 0|x^2|0\rangle}?$$

(a) $\beta = 2k^2$ (b) $\beta = k^2$ (c) $\beta = k^2/4$ (d) $\beta = k^2/2$

Ans. 4: (d)

4. Ans.(a)

Solution:

$$\langle 0|e^{ikx}|0\rangle = e^{-\beta\langle 0|x^2|0\rangle}$$

$$\Rightarrow \left\langle 0\left|\left\{1+ikx+\frac{(ikx)^2}{2!}+....\right\}\right|0\right\rangle = 1-\beta\langle 0|x^2|0\rangle+....$$

$$\Rightarrow \langle 0|1|0\rangle + ik\langle 0|\hat{x}|0\rangle - \frac{k^2}{2}\langle 0|x^2|0\rangle + ... = 1-\beta\langle 0|x^2|0\rangle$$

$$\Rightarrow 1+0-\frac{k^2}{2}\langle 0|x^2|0\rangle = 1-\beta\langle 0|x^2|0\rangle \quad \Rightarrow \frac{k^2}{2} = \beta$$

Q5. A particle of mass m moves in one dimension. The exact eigenfunctions for the ground state of the system is

$$\psi(x) = \frac{A}{\cosh(\lambda x)},$$

where, λ is a constant and A is the normalization constant. If the potential $V(x)$ vanishes at infinity, the ground state energy of the system is

(a) $-\frac{\hbar^2\lambda}{2m}$ (b) $\frac{\hbar^2\lambda^2}{2m}$ (c) $\frac{\hbar^2\lambda}{2m}$ (d) $-\frac{\hbar^2\lambda^2}{2m}$

Ans. 5: (d)

Solution:

$$\psi(x) = \frac{A}{\cosh(\lambda x)} = A\sec h(\lambda x)$$

Time independent Schrodinger equation; $-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi(x) + V\psi(x) = E\psi(x)$

$$-\frac{\hbar^2}{2m}\frac{d}{dx}\frac{d}{dx}(\sec h(\lambda x)) + V\sec h(\lambda x) = E\sec h(\lambda x) \qquad ....(1)$$

Now $\frac{d}{dx}\left(\sec\left(\lambda x\right)\right) = \lambda \tanh\left(\lambda x\right)\sec h\left(\lambda x\right)$

and $\frac{d}{dx}\left[\lambda \tanh\left(\lambda x\right)\sec\left(\lambda x\right)\right] = \lambda \frac{d}{dx}\left(\tanh\left(\lambda x\right)\right)\sec\left(\lambda x\right) + \tanh\left(\lambda x\right)\frac{d}{dx}\left(\sec h\left(\lambda x\right)\right)$

$$= \lambda^2 \sec h^2\left(\lambda x\right)\sec\left(\lambda x\right) + \lambda \tanh\left(\lambda x\right)\left(\lambda \tanh\left(\lambda x\right)\sec\left(\lambda x\right)\right)$$

$$= \lambda^2 \sec\left(\lambda x\right)\left[\sec h^2\left(\lambda x\right) + \tanh^2\left(\lambda x\right)\right] = \lambda^2 \sec h\left(\lambda x\right)$$

$\therefore$ From (1), $-\frac{\hbar^2}{2m}\left[\lambda^2 \sec h\left(\lambda x\right)\right] + V \sec h\left(\lambda x\right) = E \sec h\left(\lambda x\right) \quad \Rightarrow -\frac{\hbar^2\lambda^2}{2m} + V = E$

Now at $x \to \infty$, $V = 0$; $\quad \therefore E = -\frac{\hbar^2\lambda^2}{2m}$

Thus correct option is (d).

Part-C: 2-Mark Numerical Questions

No Question

Part-A: 1-Mark Questions

Q5. A system with two energy levels is in thermal equilibrium with a heat reservoir at temperature $600\,K$. The energy gap between the levels is $0.1eV$. Let p be the probability that the system is in the higher energy level. Which of the following statement is correct? [Note: $1eV \simeq 11600\,K$]

(a) $0 < p \le 0.1$ (b) $0.1 < p \le 0.2$ (c) $0.2 < p \le 0.3$ (d) $p \ge 0.3$

Ans. 5: (b)

Solution: $T = 600K$, Note that $K_B = 8.617\times10^{-5}$ eV/K

$$P = \frac{e^{-\beta\varepsilon}}{1+e^{-\beta\varepsilon}} = \frac{1}{1+e^{\beta\varepsilon}} = \frac{1}{1+e^{\frac{0.1\text{eV}}{8.617\times10^{-5}eV}\times600}}$$

$$= \frac{1}{1+e^{\frac{0.1}{0.0517}}} = \frac{1}{1+6.917} \approx 0.126 \approx 0.13$$

Q6. If mean and standard deviation of the energy distribution of an equilibrium system vary with temperature T as T^{ν} and T^{α} respectively, then ν and α must satisfy

(a) $2\nu = 1+\alpha$ (b) $2\nu+1=\alpha$ (c) $\nu = 1+2\alpha$ (d) $\nu+1 = 2\alpha$

Ans. 6: (d)

Solution: Consider a system of classical ideal gas with N molecules

Mean energy $U = N\langle E\rangle = \frac{3}{2}Nk_BT \propto T^1$, $\quad \therefore \nu = 1$

For this system, standard deviation in energy i.e. $\sigma_E = \sqrt{(\Delta E)^2} = \sqrt{kT^2C_V} \propto T^1$, $\therefore \alpha = 1$

$\because \nu+1 = 2\alpha \qquad \Rightarrow 1+1 = 2\times1$

Q7. Adding $1eV$ of energy to a large system did not change its temperature $\left(27^{\circ}C\right)$ whereas it changed the number of micro-states by a factor r.

r is of the order [Note: $1eV \simeq 11600\,K$]

(a) 10^4 (b) 10^{23} (c) 10^{17} (d) 10^{-19}

Ans. 7: (c)

Solution: $\because \beta = \left(\frac{\partial \ln\Omega}{\partial E}\right)$, $\quad \Omega$ being the number of microstates in the system.

This allows us to write, $\Delta \ln\Omega = \beta\Delta E = \frac{\Delta E}{k_BT}$

$\therefore$ by changing energy by ΔE, Ω will change by a factor of $e^{\frac{\Delta E}{kT}} = e^{\frac{1\text{eV}}{25\text{meV}}} \approx 2.3\times10^{17}$

Q8. The ratio of specific heat of electrons in a heated copper wire at two temperatures $200^{\circ}C$ and $100^{\circ}C$ is

(a) 1.27 (b) 2 (c) 1.41 (d) 1.61

Ans. 8: (a)

Solution:

Total heat capacity

$$C_{VT} = \underbrace{aT}_{\text{electronic contribution}} + \underbrace{bT^3}_{\text{Lattice contribution}}$$

$\therefore C_{V\,\text{electronic}} \propto T$ i.e. $C_{Ve1} = aT_1$ and $C_{Ve_2} = aT_2$

$$\therefore \frac{C_{Ve1}}{C_{Ve2}} = \frac{T_1}{T_2} = \frac{473}{373} = 1.27$$

Q21. An ideal diatomic gas at pressure P is adiabatically compressed so that its volume becomes $\frac{1}{n}$ times the initial value. The final pressure of the gas will be

(a) $n^{\frac{7}{5}}P$ (b) $n^{\frac{7}{2}}P$ (c) $n^{-\frac{7}{5}}P$ (d) $n^{\frac{5}{3}}P$

Ans. 21: (a)

Solution:

$V_1 = V$; $V_2 = \frac{V}{n}$, for a diatomic gas $C_V = \frac{5}{2}R,\ C_P = \frac{7}{2}R$ $\therefore r = \frac{C_P}{C_V} = \frac{7}{5}$

Now for an adiabatic compression $P_1V_1^r = P_2V_2^r \Rightarrow P_2 = \left(\frac{V_1}{V_2}\right)^{\gamma} P_1 = \left(\frac{V}{V/n}\right)^{\frac{7}{5}} P = n^{\frac{7}{5}}P$

Part-B: 3-Mark Questions

Q7. The energy of two Ising spins $(s_1 = \pm 1, s_2 = \pm 1)$ is given by $E = -s_1 s_2 - \frac{1}{2}(s_1 + s_2)$. At certain temperature T probability that both spins take $+1$ values is 4 times than they both take -1 values. What is the probability that they have opposite spins? $\left[\beta = 1/k_B T\right]$

(a) $\dfrac{e^{\beta}}{1+e^{2\beta}}$ (b) $e^{\beta}\tanh\beta$ (c) $\dfrac{1}{6}$ (d) $\dfrac{1}{2}$

Ans. 7: (c)

Solution:

Given $E = -s_1 s_2 - \frac{1}{2}(s_1 + s_2)$

s_1	s_2	E
+1	+1	-2
+1	-1	+1
-1	+1	+1
-1	-1	0

$$P(+1,+1) = \frac{e^{2\beta}}{1+e^{2\beta}+2e^{-\beta}}$$

$$P(-1,-1) = \frac{e^{0}}{1+e^{2\beta}+2e^{-\beta}} = \frac{1}{1+e^{2\beta}+2e^{-\beta}}$$

Given that $P(+1,+1) = 4P(-1,-1)$ i.e.

$$\frac{e^{2\beta}}{1+e^{2\beta}+2e^{-\beta}} = \frac{4}{1+e^{2\beta}+2e^{-\beta}} \qquad \Rightarrow e^{2\beta} = 4 \Rightarrow e^{\beta} = 2$$

$$\therefore P(+1,-1) = \frac{2e^{-\beta}}{1+e^{2\beta}+2e^{-\beta}} = \frac{2\times\frac{1}{2}}{1+2^2+2\times\frac{1}{2}} = \frac{1}{6}$$

Q15. A container has two compartments. One compartment contains Oxygen gas at pressure P_1, volume V_1 and temperature T_1. The second compartment contains Nitrogen gas at pressure P_2, volume V_2, and temperature T_2. The partition separating two compartments is removed and the gases are allowed to mix. What is the temperature of the mixture when it comes to equilibrium?

(a) $\dfrac{(P_1V_1+P_2V_2)T_1T_2}{P_1V_1T_2+P_2V_2T_1}$ (b) $\dfrac{(V_1T_1+V_2T_2)}{V_1+V_2}$ (c) $\dfrac{(P_1V_1T_2+P_2V_2T_1)}{P_1V_1+P_2V_2}$ (d) $\dfrac{(P_1V_1T_1+P_2V_2T_1)}{P_1V_1+P_2V_2}$

Ans. 15: (a)

Solution:

$\because$ Gases are identical (diatomic) $\qquad C_{V1} = C_{V2} = \frac{5}{2}R$ per mole

$$\boxed{\begin{matrix} n_1, T_1, C_V \\ O_2 \end{matrix}} + \boxed{\begin{matrix} n_2, T_2, C_V \\ N_2 \end{matrix}} \Rightarrow \boxed{\begin{matrix} (n_1+n_2), T(T_1, T_2) \\ C_V \end{matrix}}$$

$$U_1 = n_1 C_V T_1;\ U_2 = n_2 C_V T_2; \qquad U_T = (n_1+n_2) C_V T$$

$$\therefore\ U_T = U_1 + U_2 \quad \Rightarrow (n_1+n_2) C_V T = n_1 C_V T_1 + n_2 C_V T_2$$

$$\Rightarrow T = \frac{n_1 T_1 + n_2 T_2}{n_1 + n_2}, \qquad \because n_1 = \frac{P_1 V_1}{R T_1}, n_2 = \frac{P_2 V_2}{R T_2}$$

$$\Rightarrow T = \frac{\frac{P_1 V_1}{R T_1} \times T_1 + \frac{P_2 V_2}{R T_2} T_2}{\frac{P_1 V_1}{R T_1} + \frac{P_2 V_2}{R T_2}} = \frac{P_1 V_1 + P_2 V_2}{\frac{P_1 V_1}{T_1} + \frac{P_2 V_2}{T_2}} = \frac{(P_1 V_1 + P_2 V_2) T_1 T_2}{P_1 V_1 T_2 + P_2 V_2 T_1}$$

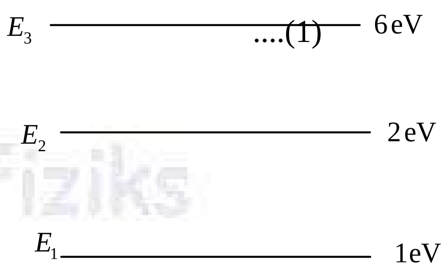
Part-C: 2-Mark Numerical Questions

Q4. A particle can access only three energy levels $E_1 = 1eV, E_2 = 2eV$, and $E_3 = 6eV$. The average energy $\langle E \rangle$ of the particle changes as temperature T changes. What is the ratio of the minimum to the maximum average energy of the particle?

Ans.: 0.333

Solution:

$$\theta = e^{-\beta} + e^{-2\beta} + e^{-6\beta} \quad ....(1)$$



$$\langle E \rangle = -\frac{\partial \ln \theta}{\partial \beta}$$

$$\langle E \rangle = \frac{1e^{-\beta} + 2e^{-2\beta} + 6e^{-6\beta}}{e^{-\beta} + e^{-2\beta} + e^{-6\beta}}$$

Now minimum $\langle E \rangle_{\min}$ is obtained when particle is in level E_1

$$\langle E \rangle_{\min} = \frac{E_1 e^{-\beta}}{\theta} = \frac{e^{-\beta}}{\theta}; \qquad \langle E \rangle_{\max} = \frac{E_3 e^{-6\beta}}{\theta} = \frac{6e^{-6\beta}}{\theta}$$

$$\frac{\langle E_{\min} \rangle}{\langle E_{\max} \rangle} = \frac{e^{-\beta}}{\theta} \times \frac{\theta}{6e^{-6\beta}} = \frac{1}{6e^{-5\beta}}$$

It seems more information is needed to evaluate the ratio.

Q5. A system of N classical non-identical particles moving in one dimensional space is governed by the Hamiltonian

$$H = \sum_{i=1}^{N}\left(A_i p_i^2 + B_i |q_i|^{\alpha}\right)$$

where p_i and q_i are momentum and position of the i-th particle, respectively, and the constant parameters $\mathrm{A_i}$ and B_i characterize the individual particles. When the system is in equilibrium at temperature T, then the internal energy is found to be

$$E = \langle H \rangle = \frac{2}{3} N k_B T,$$

where k_B is the Boltzmann constant. What is the value of α ?

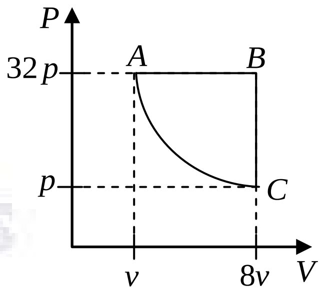
Ans.: 6

Solution:

Average energy for N particles is $\langle E \rangle = N\left[\frac{1}{2}k_B T + \frac{k_B T}{\alpha}\right] = \frac{2}{3} N k_B T \quad \Rightarrow \frac{1}{2} + \frac{1}{\alpha} = \frac{2}{3}$

$$\Rightarrow \frac{\alpha+2}{2\alpha} = \frac{2}{3} \Rightarrow 3(\alpha+2) = 4\alpha \quad \Rightarrow \boxed{\alpha = 6}$$

Q10. One mole of an ideal gas undergoes a thermodynamic cycle formed by an isobaric process, an isochoric process, and an adiabatic process (see figure). At A, the temperature of the gas is T. What is the change in the internal energy of the gas, in the units of RT (R is the universal gas constant) as the system goes from A to B

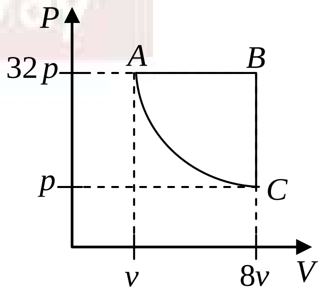


Ans.: 17.5

Solution:

$n = 1,\ PV = RT$ ....(1)

$\Delta U_{AB} = ?$



$\Delta U_{\text{Total}} = \Delta U_{AB} + \Delta U_{BC} + \Delta U_{CA} = 0$

$\Rightarrow \Delta U_{AB} = -\left[\Delta U_{BC} + \Delta U_{CA}\right]$

$B \to C$ Isochoric process $\Rightarrow dV = 0$

$\therefore \Delta U_{BC} = \Delta Q_{BC} - dW = \Delta Q_{BC} - 0 = \Delta Q_{BC} = C_V\left[T_C - T_B\right]$

Process $C \to A$ is adiabatic, i.e. $\Delta Q_{CA} = 0$; $\quad \Delta U_{CA} = \Delta Q_{CA} - \Delta W_{CA} = -\Delta W_{CA}$

$$\Delta W_{CA} = \frac{R}{\gamma - 1}[T_C - T_A] = \frac{R}{1-\gamma}[T_A - T_C]$$

Using $P_A V_A^\gamma = P_C V_C^\gamma \Rightarrow 32PV^\gamma = P(8V)^\gamma \Rightarrow 32V^\gamma = 8^\gamma V^\gamma \Rightarrow \ln 32 = \ln 8^\gamma$

$$\Rightarrow \ln 2^5 = \gamma \ln 2^3 \Rightarrow 5\ln 2 = 3\gamma \ln 2 \quad \Rightarrow \boxed{\gamma = 5/3} \Rightarrow \text{monoatomic gas} \Rightarrow C_V = \frac{3}{2}R$$

$$\Delta W_{CA} = \frac{R}{1-\frac{5}{3}}[T_A - T_C] = -\frac{3R}{2}[T_A - T_C]$$

$$\Rightarrow -\Delta W_{CA} = \frac{3R}{2}[T_A - T_C] = \Delta U_{CA} \text{ and } \Delta U_{BC} = \frac{3}{2}R[T_C - T_B]$$

$$\therefore \Delta U_{AB} = -\Delta U_{BC} - \Delta U_{CA} = -\cancel{\frac{3}{2}RT_C} + \frac{3}{2}RT_B - \frac{3RT_A}{2} + \cancel{\frac{3}{2}RT_C} \quad \Rightarrow \boxed{\Delta U_{AB} = \frac{3R}{2}[T_B - T_A]}$$

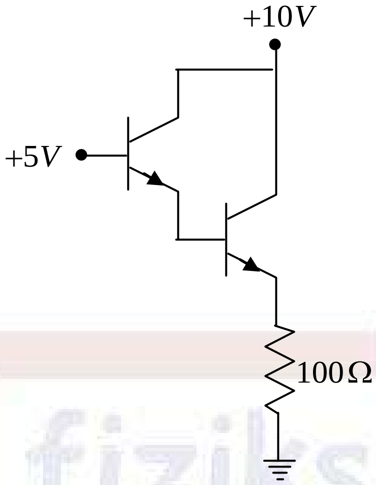
Now for process AB (Isobaric) ; $PV = RT$

$$\Rightarrow T_B = \frac{32P \times 8V}{R} \text{ and } T_A = \frac{32PV}{R}$$

$$\therefore \Delta U_{AB} = \frac{3R}{2}\left[32 \times 8\frac{PV}{R} - \frac{32PV}{R}\right] = \frac{3}{2}R \times 32 \times \frac{7PV}{R} = 336PV \quad \Rightarrow \boxed{\Delta U_{AB} = 336RT}$$

Part-A: 1-Mark Questions

Q20. The base current in the first transistor of the following circuit having two identical Silicon-based *npn* transistors of β value 100, is closest to



(a) $3.6\,\mu A$ (b) $0.36\,mA$ (c) $5.0\,mA$ (d) $5.0\,\mu A$

Ans. 20: (a)

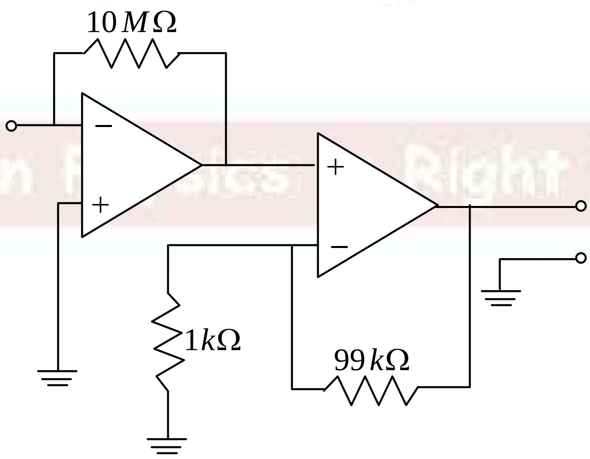
Solution.:

$$I_{B_1} = \frac{V_{CC} - V_{BE_1} - V_{BE_2}}{\beta_D R_E} \text{ where } \beta_D = \beta_1.\beta_2 = 100\times100 = 10^4.$$

$$\Rightarrow I_{B_1} = \frac{5-0.7-0.7}{10^4\times100} = 3.6\mu A$$

Part-B: 3-Mark Questions

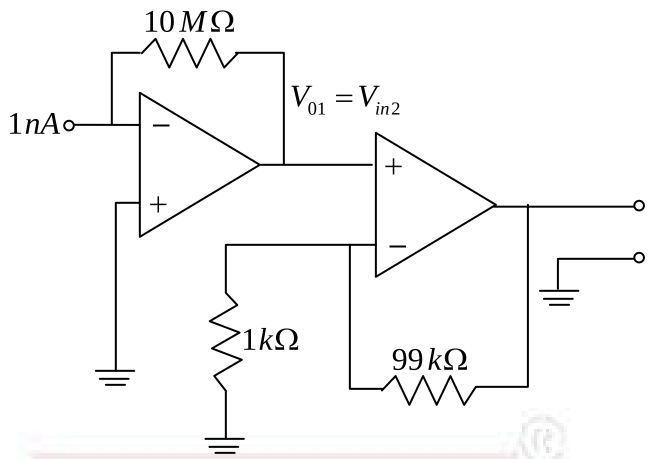
Q14. What is the output voltage of the following circuit for the input current $1nA$?



(a) $1mV$ (b) $1V$ (c) $1\mu V$ (d) $1nV$

Ans. 14: (b)

Solution.:



$$\Rightarrow V_{01} = V_{in2} = 1nA \times 10M\Omega = 1\times10^{-9}\times10\times10^{6} = 0.01\,Volts$$

$$\Rightarrow V_0 = \left(1+\frac{R_F}{R_1}\right)V_{in2} = \left(1+\frac{99}{1}\right)\times 0.01 = 1\,Volts$$

Part-C: 2-Mark Numerical Questions

Q3. Optical excitation of intrinsic germanium creates an average density of 10^{12} conduction electrons per cm^3 in the material at liquid nitrogen temperature. At this temperature, the electron and hole nobilities are equal, $\mu = 0.5\times10^4\, m^2V^{-1}s^{-1}$. The germanium dielectric constant is 20. If 100 Volts is applied across $1cm$ cube of crystal under these conditions, about how much current, in mA, is observed? [Charge of electron $=1.6\times10^{-19}C$]

Ans.: 0.08

Q9. A 12-bit analog-to-digital converter has an operating range of 0 to $1V$. The smallest voltage step (in mV, upto two significant digits) that one can record using this converter is

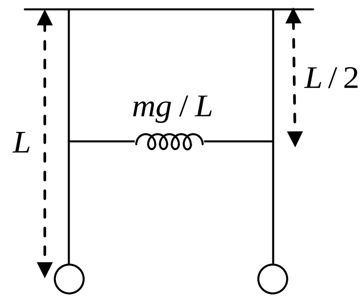
Ans.: 0.24

Solution.:

$$\text{Step size} = \frac{1V}{2^{12}-1} = \frac{1}{4096-1}V = 0.24\,mV$$

Part-A: 1-Mark Questions

Q16. Two identical simple pendulum of length L are connected by a spring at a height of $L/2$ as shown in the figure. Assuming the spring constant is mg/L, where m is the mass of the bob and g is the acceleration due to gravity, what is the ratio of the highest to lowest Eigen frequencies of the system?

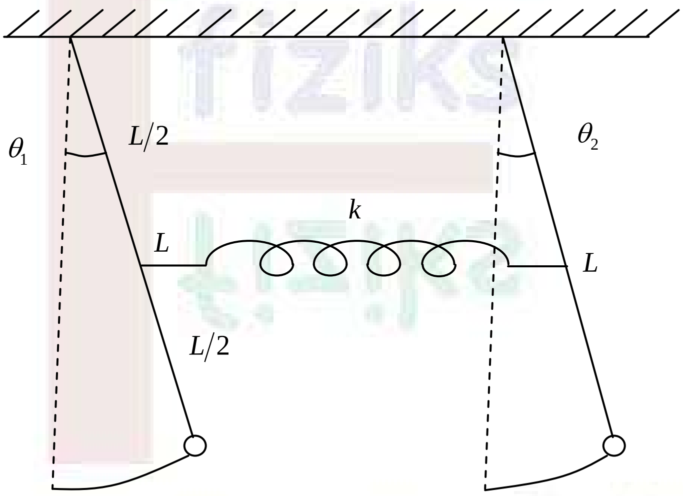


(a) 1 (b) $\sqrt{3/2}$

(c) $\sqrt{2}$ (d) $\sqrt{3}$

Ans. 16: (b)

Solution:



$$T = \frac{1}{2}mL^2\dot{\theta}_1^2 + \frac{1}{2}mL^2\dot{\theta}_2^2; \quad T = \begin{pmatrix} mL^2 & 0 \\ 0 & mL^2 \end{pmatrix}$$

$$V = mgL\left(1-\cos\theta_1\right) + mgL\left(1-\cos\theta_2\right) + \frac{1}{2}k\left(\frac{L}{2}\theta_1 - \frac{L}{2}\theta_2\right)^2$$

$$= 2mgL\sin^2\frac{\theta_1}{2} + 2mgL\sin^2\frac{\theta_2}{2} + \frac{1}{2}k\frac{L^2}{4}\left(\theta_1^2 + \theta_2^2 - 2\theta_1\theta_2\right)$$

$$= \frac{1}{2}\left(mgL + \frac{1}{4}kL^2\right)\theta_1^2 + \frac{1}{2}\left(mgL + \frac{1}{4}kL^2\right)\theta_2^2 - \frac{1}{2}\frac{kL^2}{4}\left(\theta_1\theta_2 + \theta_2\theta_1\right)$$

$$V = \begin{bmatrix} mgL + \frac{1}{4}kL^2 & -\frac{1}{4}kL^2 \\ -\frac{1}{4}kL^2 & mgL + \frac{1}{4}kL^2 \end{bmatrix}$$

$$\left|V - \omega^2 T\right| = 0$$

$$\Rightarrow\begin{vmatrix} mgL+\frac{1}{4}kL^2-\omega^2mL^2 & -\frac{1}{4}kL^2 \\ -\frac{1}{4}kL^2 & mgL+\frac{1}{4}kL^2-\omega^2mL^2 \end{vmatrix}=0$$

$$\Rightarrow\left(mgL+\frac{1}{4}kL^2-\omega^2mL^2\right)^2-\left(\frac{1}{4}kL^2\right)^2=0$$

$$\Rightarrow\left(mgL+2\cdot\frac{1}{4}kL^2-\omega^2mL^2\right)\left(mgL-\omega^2mL^2\right)=0$$

$$\omega_1^2=\frac{mgL+\frac{2}{4}kL^2}{mL^2} \text{ and } \omega_2^2=\frac{mgL}{mL^2}$$

$$\frac{\omega_1^2}{\omega_2^2}=\frac{mgL+2k\frac{L^2}{4}}{mgL}=1+\frac{2kL}{4mg}=1+\frac{1}{2}\left(\frac{mg}{L}\right)\frac{L}{mg} \qquad \because k=\frac{mg}{L}$$

$$\boxed{\frac{\omega_1}{\omega_2}=\sqrt{\frac{3}{2}}}$$

Q18. A thin film surrounded by air has an index of refraction of 1.4. A region of the film appears bright blue $(\lambda=400\,nm)$ when white light is incident perpendicular to the surface. What might be the minimum thickness of the film?

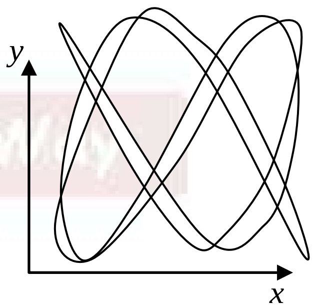
(a) $420\,nm$ (b) $280\,nm$ (c) $140\,nm$ (d) $70\,nm$

Ans. 18: (c)

Solution:

$$2\mu t=\lambda\Rightarrow t=\frac{\lambda}{2\mu}=\frac{400\,nm}{2\times1.4}=142.84\,nm$$

Q19. The trajectory of a particle which undergoes simple harmonic motion on a plane is shown in the figure. The ratio of the frequencies for the motion along x and y directions is given by



(a) $\frac{4}{5}$ (b) $\frac{2}{3}$

(c) $\frac{3}{2}$ (d) $\frac{3}{5}$

Ans. 19: (d)

Solution:

$$n_x=10, n_y=6;\quad \frac{\omega_x}{\omega_y}=\frac{n_y}{n_x}=\frac{6}{10}=\frac{3}{5}$$

Part-B: 3-Mark Questions

No Question

Part-C: 2-Mark Numerical Questions

Q2. A pair of crossed ideal linear polarizer's allow no light to pass through. To produce some output one can insert optical elements between the crossed polarizer's. For given light beam of input intensity I_0 Nirmalya inserts a quarter-wave plate between the crossed polarizer's and records an output intensity αI_0. On the other hand, Ayan inserts two linear polarizer's having orientations 30° and 60° w.r.t. the first polarizer of the crossed pair, and records an output intensity of βI_0. What is the ratio $\dfrac{\alpha}{\beta}$?

Ans.: 1.19

Solution: $I_1 = \left(\left(\dfrac{I_0}{2}\cos^2 30^\circ\right)\cos^2 30^\circ\right)\cos^2 30^\circ = \left(\left(\dfrac{I_0}{2}\cdot\dfrac{3}{4}\right)\dfrac{3}{4}\right)\dfrac{3}{4} = \dfrac{27I_0}{128}$

$$I_1 = \beta I_0 \Rightarrow \beta = \frac{27}{128}$$

$$I_2 = \frac{I_0}{2}\sin^2 2\theta \sin^2\frac{\delta}{2} = \frac{I_0}{2}\cdot\frac{1}{2}\sin^2 2\theta \qquad \left[\because \delta = \frac{\pi}{2}\right]$$

For maximum intensity $\sin^2 2\theta = 1 \rightarrow \theta = 45^\circ$

$$I_2 = \frac{I_0}{4} = \alpha I_0 \Rightarrow \alpha = \frac{1}{4}$$

$$\frac{\alpha}{\beta} = \frac{1}{4}\cdot\frac{128}{27} = \frac{128}{108} = 1.185 \simeq 1.19$$

Q8. The frequency dispersion relation of the surface waves of a fluid of density ρ and temperature T, is given by $\omega^2 = gk + Tk^3/\rho$, where ω and k are the angular frequency and wavenumber, respectively, g is the acceleration due to gravity. The first term in r.h.s. describes the gravity waves and the second term describes the surface tension wave. What is the ratio of the first term to the second term, when the phase velocity is equal to the group velocity?

Ans.: 1

Solution:

$$\omega^2 = gk + \frac{Tk^3}{\rho}$$

$$2\omega\frac{d\omega}{dk} = g + \frac{3Tk^2}{\rho} \qquad \text{....(1)}$$

$$\omega\cdot\frac{\omega}{k} = g + \frac{Tk^2}{\rho} \qquad \text{....(2)}$$

$$2\cdot\frac{v_g}{v_p} = \frac{g + 3Tk^2/\rho}{g + Tk^2/\rho}$$

Given $v_g = v_p$; $\quad \Rightarrow 2g + 2\dfrac{Tk^2}{\rho} = g + 3\dfrac{Tk^2}{\rho} \Rightarrow g = \dfrac{Tk^2}{\rho} \Rightarrow \dfrac{gk}{Tk^3/\rho} = 1$

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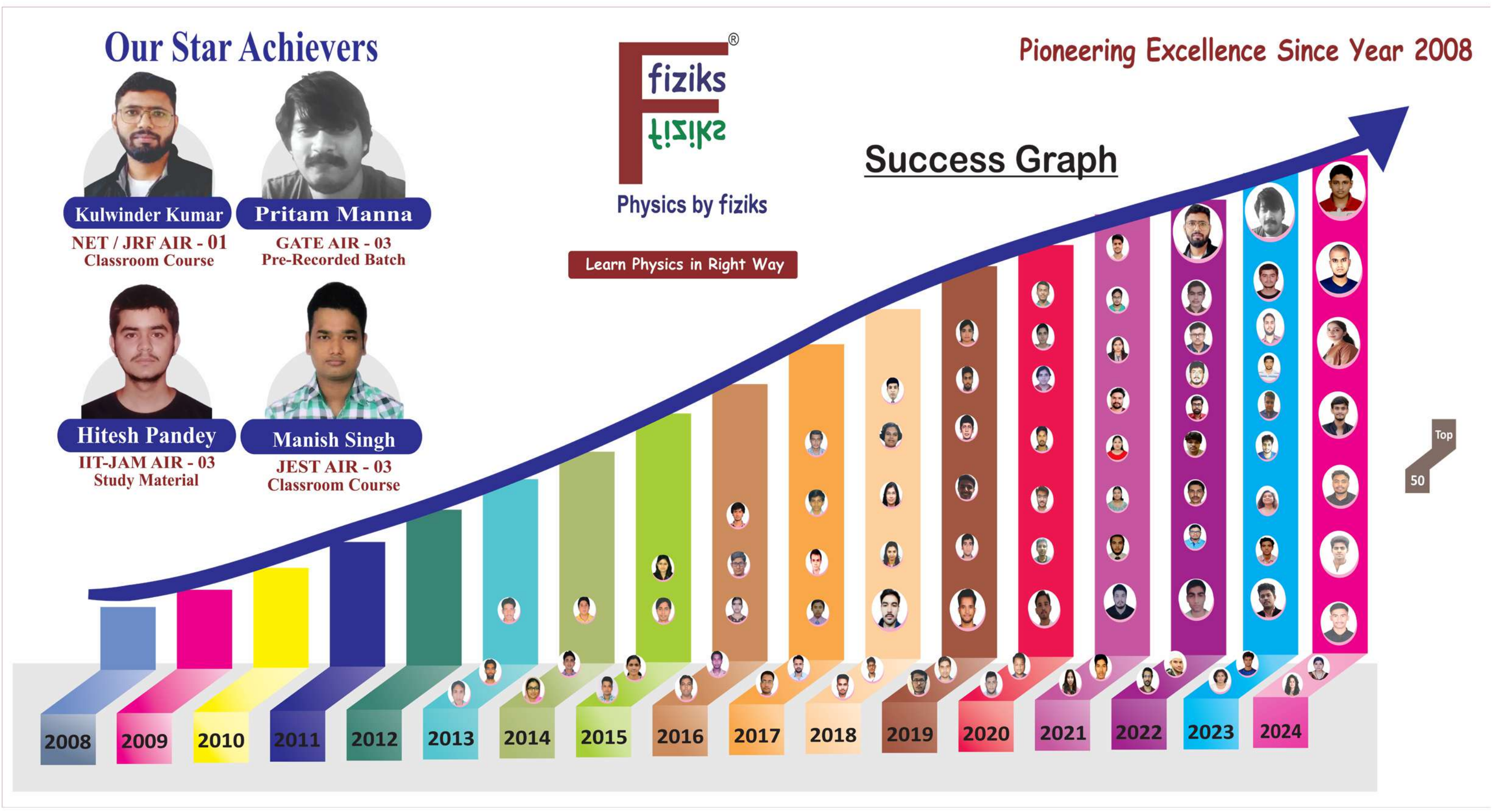

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