Physics by fiziks

Learn Physics in Right Way

JEST Physics-2021

Solution

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Part-A: 1-Mark Questions

Q2. The six faces of a cube are painted violet, blue, red, green, yellow and orange. If the cube is rolled 4 times, what is the probability that the green face appears exactly 3 times?

(a) $\frac{3}{24}$ (b) $\frac{5}{124}$ (c) $\frac{5}{324}$ (d) $\frac{15}{222}$

Ans: (c)

Solution.:

Probability of occurrence of green face $=\frac{1}{6}=p$

Probability of not occurrence of green face $=\frac{5}{6}=1-p=q$

Cube is rolled 4 times, then the probability that the green face appears exactly 3 times is

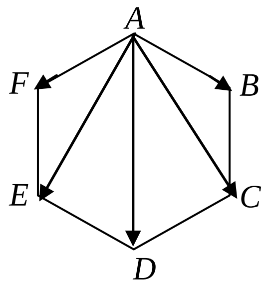
$$={}^{n}C_{r}p^{r}q^{n-r}={}^{4}C_{3}p^{3}q^{4-3}=\frac{4!}{3!1!}\left(\frac{1}{6}\right)^{3}\left(\frac{5}{6}\right)=4\left(\frac{1}{6}\right)^{3}\left(\frac{5}{6}\right)=\frac{5}{324}$$

Q7. Let ABCDEF be a regular hexagon. The vector $\overrightarrow{AB}+\overrightarrow{AC}+\overrightarrow{AD}+\overrightarrow{AE}+\overrightarrow{AF}$ will be

(a) 0 (b) $\overrightarrow{AD}$ (c) $2\overrightarrow{AD}$ (d) $3\overrightarrow{AD}$

Ans: (d)

Solution.:



$$\overrightarrow{AB}+\overrightarrow{AC}+\overrightarrow{AD}+\overrightarrow{AE}+\overrightarrow{AF}=\left(\overrightarrow{AC}+\overrightarrow{AF}\right)+\left(\overrightarrow{AB}++\overrightarrow{AE}\right)+\overrightarrow{AD}$$

$$=\left(\overrightarrow{AC}+\overrightarrow{CD}\right)+\left(\overrightarrow{ED}++\overrightarrow{AE}\right)+\overrightarrow{AD}=\overrightarrow{AD}+\overrightarrow{AD}+\overrightarrow{AD}=3\overrightarrow{AD}$$

$\because \overrightarrow{AF}=\overrightarrow{CD}$ and $\overrightarrow{ED}=\overrightarrow{AB}$

Q10. What value the following infinite series will converge to?

$$\sum_{n=1}^{\infty}\frac{n^2}{2^n}$$

(a) $\frac{\pi^2}{6}$ (b) $\frac{1}{2}$ (c) 3 (d) 6

Ans: (d)

Solution:

$S=\frac{1}{2}+\frac{4}{4}+\frac{9}{8}+\frac{16}{16}+\frac{25}{32}+...$ and $2S=1+\frac{4}{2}+\frac{9}{4}+\frac{16}{8}+\frac{25}{16}+...$

Subtracting these two will give; $S=1+\frac{3}{2}+\frac{5}{4}+\frac{7}{8}+\frac{9}{16}+\frac{1}{32}+...;$

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$$2S-2=3+=3+\frac{5}{2}+\frac{7}{4}+\frac{9}{8}+\frac{11}{16}+\ldots$$

Thus

$$2S-2-S=3+=2+\frac{2}{2}+\frac{2}{4}+\frac{2}{8}+\frac{2}{1}+\frac{}{6}+\ldots=3+\frac{2}{4}\left(1+\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\ldots\right)=3+\frac{2}{4}\left(\frac{1}{1-1/2}\right)=4$$

$$\Rightarrow S-2=4 \quad \Rightarrow S=6$$

Q22. Consider the matrix $A=\begin{pmatrix}1 & 0 & 0 & 1\\ 0 & -2 & 0 & 0\\ 0 & 0 & -3 & 0\\ 1 & 0 & 0 & 4\end{pmatrix}$

What is the determinant of the matrix $\exp(A)$?

(a) 1 (b) $\exp(24)$ (c) 24 (d) 0

Ans: (a)

Solution.: $\det e^{A}=e^{Trace\,A}=e^{0}=1 \quad \because Trace\,A=1-2-3+4=0$

Part-B: 3-Mark Questions

Q15. Consider the infinite series

$$\exp\left[\left(x+\frac{x^3}{3}+\ldots..\right)^2-\left(\frac{x^2}{2}+\frac{x^4}{4}+\ldots..\right)^2\right]$$

Which one of the following represents this series?

(a) $(1+x)^{\ln(1-x)}$ (b) $\exp\left[\sin^2 x-(\cos x-1)^2\right]$

(c) $\exp(xe^{x})$ (d) $(1-x)^{-\ln(1+x)}$

Ans: (d)

Solution.:

Let

$$z=\exp\left[\left(x+\frac{x^3}{3}+\ldots..\right)^2-\left(\frac{x^2}{2}+\frac{x^4}{4}+\ldots..\right)^2\right]\Rightarrow \ln z=\left[\left(x+\frac{x^3}{3}+\ldots..\right)^2-\left(\frac{x^2}{2}+\frac{x^4}{4}+\ldots..\right)^2\right]$$

$$\Rightarrow \ln z=\left[\left\{\left(x+\frac{x^3}{3}+\ldots..\right)+\left(\frac{x^2}{2}+\frac{x^4}{4}+\ldots..\right)\right\}\left(x+\frac{x^3}{3}+\ldots..\right)-\left(\frac{x^2}{2}+\frac{x^4}{4}+\ldots..\right)\right]$$

$$\because a^2-b^2=(a+b)(a-b)$$

$$\Rightarrow \ln z=\left(x+\frac{x^2}{2}+\frac{x^3}{3}+\frac{x^4}{4}\ldots..\right)\left(x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}\ldots..\right)=-\ln(1-x)\ln(1+x)$$

$$\Rightarrow \ln z=\ln(1-x)^{-\ln(1+x)}\Rightarrow z=(1-x)^{-\ln(1+x)}$$

Part-C: 2-Mark Numerical Questions

Q1. Consider a real tensor T_{ijk} with $i, j, k = 1, ..., 5$. It has the following properties:

$$T_{ijk} = T_{jik} = T_{ikj}, \qquad \sum_i T_{iik} = 0$$

What is the number of independent real components of this tensor?

Ans: 0030

Solution:

Number of independent real components of this tensor $= n(n+1) = 5\times 6 = 30$

Q7. Evaluate the integral to the nearest integer

$$I = 100\int_0^\infty \frac{dt}{t}\left[\exp(-t) - \exp(-10t)\right]$$

Ans: 0230

Solution.:

$$L\left[\frac{1}{t}\{\exp(-t) - \exp(-10t)\}\right] = \int_s^\infty \left(\frac{1}{\bar{s}+1} - \frac{1}{\bar{s}+10}\right) d\bar{s}$$

$$\because L\left[\frac{f(t)}{t}\right] = \int_s^\infty f(\bar{s})\, d\bar{s}$$

$$\Rightarrow L\left[\frac{1}{t}\{\exp(-t) - \exp(-10t)\}\right] = \ln\frac{\bar{s}+1}{\bar{s}+10}\Bigg|_s^\infty = \ln\frac{1+1/\bar{s}}{1+10/\bar{s}}\Bigg|_s^\infty = 0 - \ln\frac{s+1}{s+10} = \ln\frac{s+10}{s+1}$$

$$\because L\left[\frac{1}{t}\{\exp(-t) - \exp(-10t)\}\right] = \int_0^\infty \frac{1}{t}\left(e^{-t} - e^{-10t}\right)e^{-st}dt = \ln\frac{s+10}{s+1}$$

Now put $s = 0$; $\int_0^\infty \frac{1}{t}\left(e^{-t} - e^{-10t}\right)e^{-0}dt = \ln 10 \Rightarrow I = 100\int_0^\infty \frac{1}{t}\left(e^{-t} - e^{-10t}\right)dt = 100\ln 10 = 230.3$

Part-A: 1-Mark Questions

Q6. A spaceship moves away from Earth with a relativistic speed v and fires a shuttle craft in the forward direction at a speed v relative to the spaceship. The pilot of the shuttle craft launches a probe in the forward direction at a speed v relative to the Earth? What will be the speed of the probe relative to the Earth?

(a) $3v$

(b) $\dfrac{3v}{\sqrt{1-\dfrac{v^2}{c^2}}}$

(c) $\left(\dfrac{3+v^2/c^2}{1+3v^2/c^2}\right)v$

(d) $\dfrac{2v}{1+v^2/c^2}+v$

Ans: (c)

Solution:

S $\rightarrow$ spaceship; E $\rightarrow$ Earth; C $\rightarrow$ shuttle craft; P $\rightarrow$ Probe

$v_{SE}=v;\ v_{CS}=v;\ v_{PC}=v;\ \upsilon_{PE}=?$

$$v_{CE}=\frac{v_{CS}-v_{ES}}{1-\frac{v_{CS}v_{ES}}{c^2}}=\frac{v-(-v)}{1-\frac{v(-v)}{c^2}}\Rightarrow v_{CE}=\frac{2v}{1+\frac{v^2}{c^2}}$$

$$v_{PE}=\frac{v_{PC}-v_{EC}}{1-\frac{v_{PC}v_{EC}}{c^2}}=\frac{v-\left[-\frac{2v}{1+\frac{v^2}{c^2}}\right]}{1-\frac{v\left(-\frac{2v}{1+v^2/c^2}\right)}{c^2}}=\frac{v+\frac{v^3}{c^2}+2v}{1+\frac{v^2}{c^2}}\times\frac{1}{1+\frac{2v^2/c^2}{1+v^2/c^2}}$$

$$=\frac{3v+\frac{v^3}{c^2}}{1+\frac{v^2}{c^2}+\frac{2v^2}{c^2}}=v\left[\frac{3+\frac{v^2}{c^2}}{1+\frac{3v^2}{c^2}}\right]$$

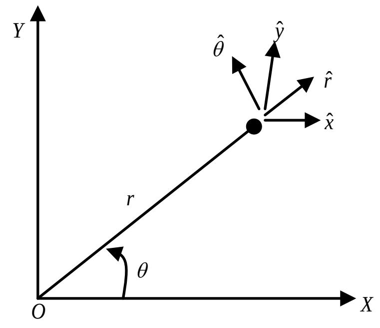
Q9. A particle of mass m is subject to the potential $V(x,y,t)=K(x^2+y^2)$, where (x,y) are the cartesian coordinates of the particle and K is a constant. Which one of the following quantities is a constant of motion?

(a) $\dot{y}x+\dot{x}y$ (b) $\dot{y}x-\dot{x}y$ (c) $\dot{y}+\dot{x}$ (d) $\dot{y}y+\dot{x}x$

Ans: (b)

Solution:

$\because L=\frac{1}{2}m\left(\dot{x}^2+\dot{y}^2\right)-k\left(x^2+y^2\right)$



If we choose polar coordinates $x=r\cos\theta$; $y=r\sin\theta$

$\Rightarrow L=\frac{1}{2}m\left(\dot{r}^2+r^2\dot{\theta}^2\right)-kr^2$

As $\frac{\partial L}{\partial\theta}=0$, so θ is the cyclic coordinate

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\theta}\right)=\frac{dp_\theta}{dt}=0$$

Conjugate momentum $p_\theta=\frac{\partial L}{\partial\dot{\theta}}=mr^2\dot{\theta}=\text{constant}$

$\vec{p}_\theta=\vec{r}\times m\left(r\dot{\theta}\right)=\vec{r}\times m\vec{v}_\theta \quad \Rightarrow \vec{p}_\theta=\vec{r}\times m\vec{v}_\theta=L_z$. So, angular momentum L_z is conserved.

$$\vec{L}=\vec{r}\times\vec{p}=m\begin{vmatrix}\hat{x} & \hat{y} & \hat{z}\\ x & y & 0\\ \dot{x} & \dot{y} & 0\end{vmatrix}\Rightarrow \vec{L}=m\left(x\dot{y}-y\dot{x}\right)\hat{z} \quad \Rightarrow L_z=m\left(x\dot{y}-y\dot{x}\right)=\text{constant}$$

Q17. A particle of mass m having a non-zero angular momentum of magnitude l is subject to a central force potential $V\left(\vec{r}\right)=k\ln\left(r\right)$, where k is a constant and $r=\left|\vec{r}\right|$. What is the radius R at which it will have a circular orbit? Will the circular orbit be stable or unstable?

(a) $R=\frac{l}{\sqrt{2km}}$, unstable orbit (b) $R=\frac{l}{\sqrt{2km}}$, stable orbit

(c) $R=\frac{l}{\sqrt{km}}$, unstable orbit (d) $R=\frac{l}{\sqrt{km}}$, stable orbit

Ans: (d)

Solution:

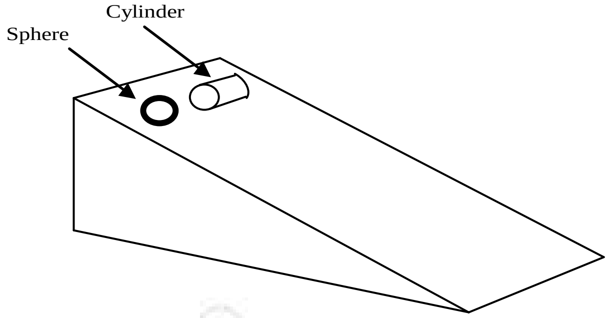
$\vec{F}=-\frac{\partial V}{\partial r}\hat{r}=-\frac{k}{r}\hat{r}$ (Attractive); $\quad V_{\text{eff}}=\frac{l^2}{2mr^2}+V\left(r\right)$

For circular orbit $\frac{\partial V_{\text{eff}}}{\partial r}=0 \Rightarrow \frac{k}{r}-\frac{l^2}{mr^3}=0 \Rightarrow r=r_0=\frac{l}{\sqrt{mk}}$

$$\frac{\partial^2V_{\text{eff}}}{\partial r^2}=-\frac{k}{r^2}+\frac{3l^2}{mr^4}\Rightarrow \left.\frac{\partial^2V_{\text{eff}}}{\partial r^2}\right|_{r=r_0}=-\frac{k}{l^2/mk}+\frac{3l^2}{ml^4/m^2k^2}=-\frac{mk^2}{l^2}+\frac{3mk^2}{l^2}$$

$\left.\frac{\partial^2V_{\text{eff}}}{\partial r^2}\right|_{r=r_0}=+\frac{2mk^2}{l^2}>0$ (Minima). So, circular orbit will stable.

Q19. A solid sphere and a solid cylinder, both of uniform mass density, start rolling down without slipping from rest from the same height along an inclined plane (see figure). Which one of the following statements is correct?



(a) The sphere would reach the bottom faster.

(b) The cylinder would reach the bottom faster.

(c) The heavier one would reach the bottom faster if both have identical radii.

(d) Both the objects would reach the bottom at the same time if their radii are identical.

Ans: (a)

Solution:

$$t=\frac{1}{\sin\theta}\sqrt{\frac{2h}{g}}\sqrt{\beta}\ ;\quad \beta=1+\frac{I}{MR^2}$$

$$\beta_S=1+\frac{\frac{2}{5}MR^2}{MR^2}=\frac{7}{5}=1.4 \text{ and } \beta_C=1+\frac{\frac{1}{2}MR^2}{MR^2}=\frac{3}{2}=1.5 \quad \Rightarrow \beta_S<\beta_C \ \Rightarrow t_S<t_C$$

So, the time taken by the sphere will be lesser to reach the bottom.

Q24. If $\vec{x}_A$ and $\vec{x}_B$ are the position vectors of two points on a rigid body, which one of the following is NOT necessarily true?

(a) $\ddot{\vec{x}}_A-\ddot{\vec{x}}_B=0$

(b) $(\vec{x}_A-\vec{x}_B)\cdot(\dot{\vec{x}}_A-\dot{\vec{x}}_B)=0$

(c) $(\vec{x}_A-\vec{x}_B)\cdot(\ddot{\vec{x}}_A-\ddot{\vec{x}}_B)+|\dot{\vec{x}}_A-\dot{\vec{x}}_B|=0$

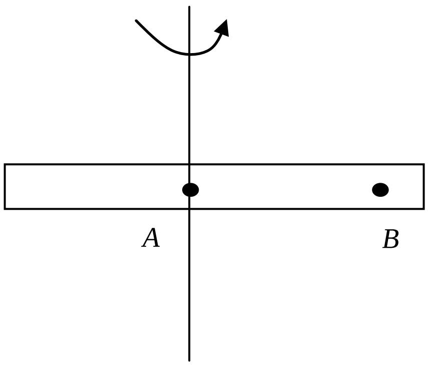
(d) $\frac{d}{dt}|\vec{x}_A-\vec{x}_B|=0$

Ans: (a)

Solution:

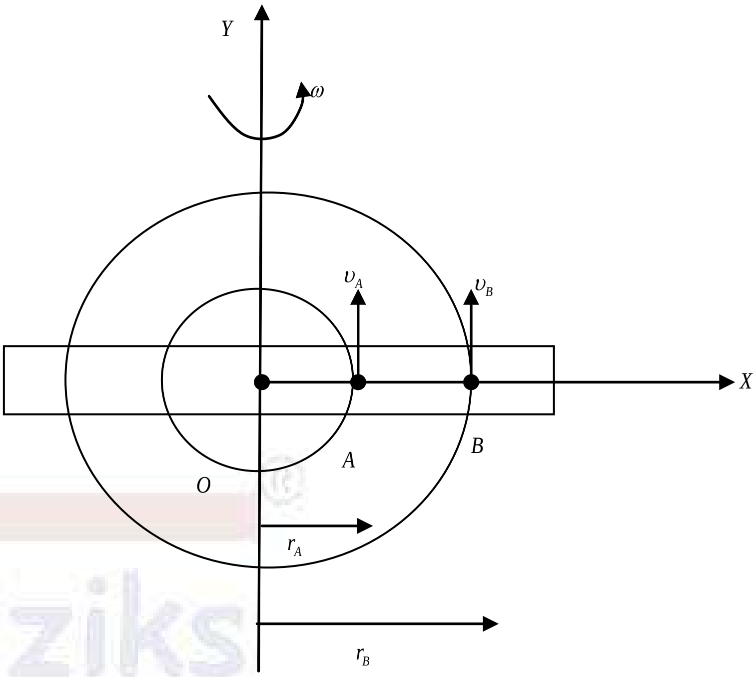
For a rigid body $|\vec{x}_A-\vec{x}_B|=c_1$

(d) $\frac{d}{dt}|\vec{x}_A-\vec{x}_B|=0$



(a) Acceleration of point A and point B is not necessarily equal.

For example rotation of a solid rod about a fixed axis.



Here $\ddot{x}_A \neq \ddot{x}_B$ because point A is at rest while B is rotating about the axis.

(b) $\vec{x}_{AB} = \vec{x}_A - \vec{x}_B$; $\vec{\upsilon}_{AB} = \dot{\vec{x}}_A - \dot{\vec{x}}_B$

Component of relative velocity along the direction of $\vec{x}_{AB}$ will always be zero in case of rigid body.

$(\vec{r}_A - \vec{r}_B)\cdot(\vec{\upsilon}_A - \vec{\upsilon}_B) = 0$

So, $(\vec{x}_A - \vec{x}_B)\cdot(\dot{\vec{x}}_A - \dot{\vec{x}}_B) = 0$

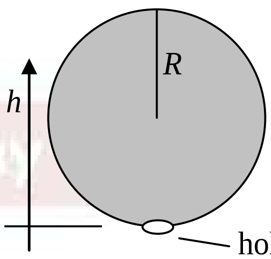
(c) In the above example, one can observe that it is not necessary that $\vec{\upsilon}_A = \vec{\upsilon}_B$; but component of relative velocity $\vec{\upsilon}_{AB} = \vec{\upsilon}_A - \vec{\upsilon}_B$ along $\vec{r}_{AB} = \vec{r}_A - \vec{r}_B$ is always zero.

So, $(\vec{x}_A - \vec{x}_B)\cdot(\dot{\vec{x}}_A - \dot{\vec{x}}_B) + \left|\dot{\vec{x}}_A - \dot{\vec{x}}_B\right|^2 = 0$ as

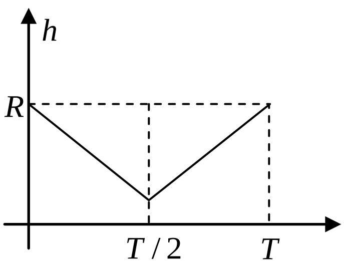
$\dot{\vec{x}}_A - \dot{\vec{x}}_B = 0$

Part-B: 3-Mark Questions

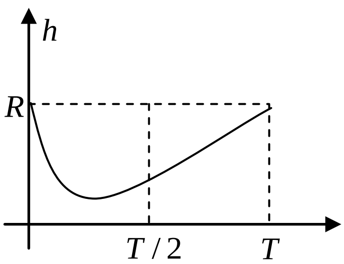
Q2. A hollow sphere of radius R, with a small hole at the bottom, is completely filled with a liquid of uniform density (see figure). The liquid drains out of the sphere through the hole at an uniform rate in time T. Which one of the following graphs (a, b, c, d) qualitatively represents the height h of the center of mass (of sphere + liquid inside it), measured from the bottom of the sphere with time?



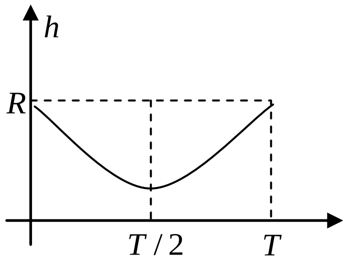
(a)



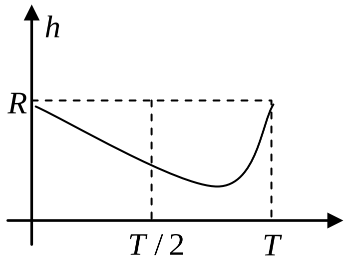
(b)



(c)



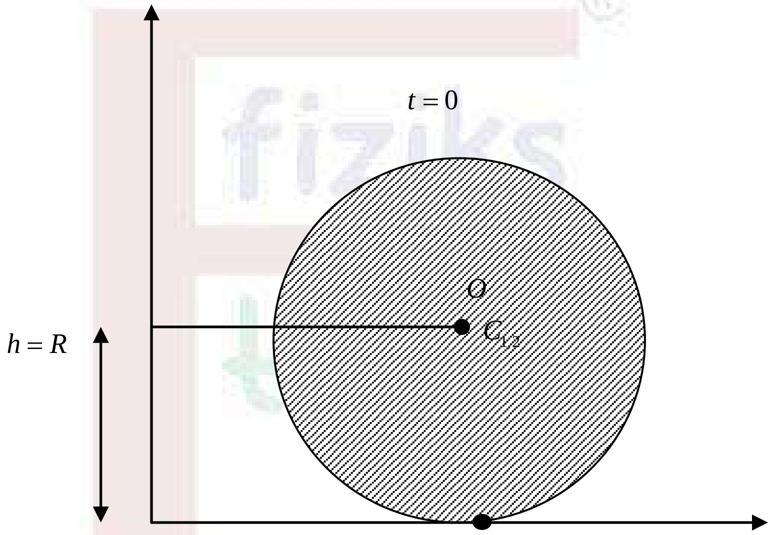
(d)



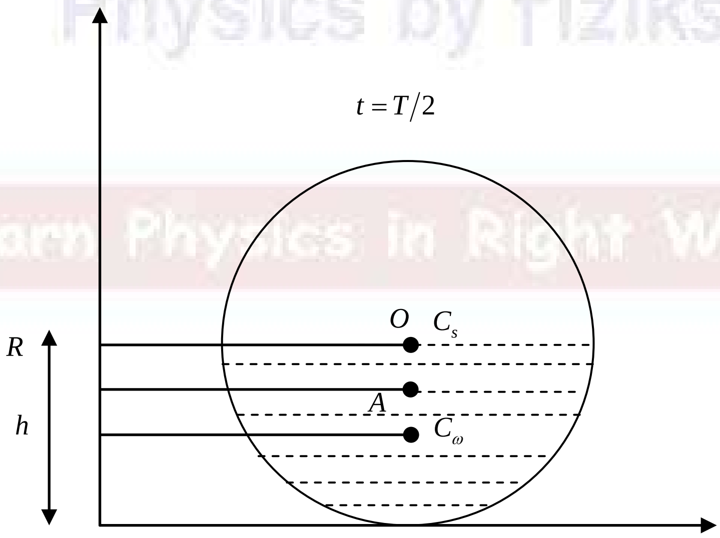
Ans: (d)

Solution: Water is draining at a constant rate.

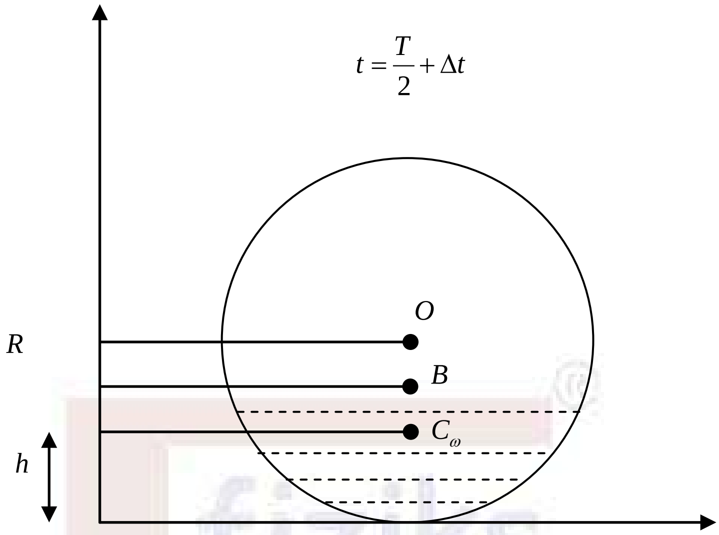
Here CM of sphere coincides with CM of water at O at time $t = 0$.



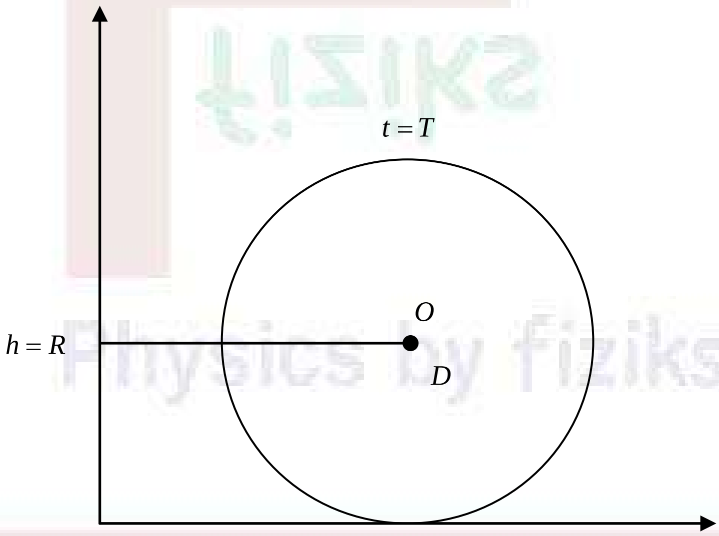
Here CM of the system (A) will be just below the centre of sphere, at $t = T/2$.



Here CM of the system (B) go down upto a particular time and after that it will start moving up.



At time $t = T$, when sphere will be completely empty, the CM of the system will be at the centre of the sphere (O).



In the light of four above cases, option (d) is correct.

Q12. Which one of the following sets correctly represents the Hamilton's equations of motion obtained from the Lagrangian $L = \frac{1}{2}m\dot{x}\dot{y} - \frac{1}{2}m\omega^2 xy$

(a) $m\dot{x} = 2p_y,\ \dot{p}_x = -\frac{1}{2}m\omega^2 y,\ m\dot{y} = 2p_x,\ \dot{p}_y = -\frac{1}{2}m\omega^2 x$

(b) $m\dot{x} = 2p_y,\ \dot{p}_x = -\frac{1}{2}m\omega^2 x,\ m\dot{y} = 2p_x,\ \dot{p}_y = -\frac{1}{2}m\omega^2 y$

(c) $m\dot{x} = p_x,\ \dot{p}_x = -m\omega^2 x,\ m\dot{y} = p_y,\ \dot{p}_y = -m\omega^2 y$

(d) $m\dot{x} = p_y,\ \dot{p}_x = -m\omega^2 y,\ m\dot{y} = p_x,\ \dot{p}_y = -m\omega^2 x$

Ans: (a)

Solution:

$$p_x = \frac{\partial L}{\partial \dot{x}} = \frac{1}{2}m\dot{y} \Rightarrow m\dot{y} = 2p_x \Rightarrow \dot{y} = \frac{2p_x}{m}$$

$$p_y = \frac{\partial L}{\partial \dot{y}} = \frac{1}{2}m\dot{x} \Rightarrow m\dot{x} = 2p_y \Rightarrow \dot{x} = \frac{2p_y}{m}$$

$$H = \dot{x}p_x + \dot{y}p_y - L = \frac{2p_xp_y}{m} + \frac{2p_xp_y}{m} - L = \frac{4p_xp_y}{m} - \frac{1}{\cancel{2}}\cancel{m}\frac{\cancel{2}p_y}{\cancel{m}}\frac{2p_x}{m} + \frac{1}{2}m\omega^2 xy$$

$$\Rightarrow H = \frac{2p_xp_y}{m} + \frac{1}{2}m\omega^2 xy$$

$$\dot{x} = \frac{\partial H}{\partial p_x} \Rightarrow \dot{x} = \frac{2p_y}{m} \Rightarrow \boxed{m\dot{x} = 2p_y}$$

$$\dot{y} = \frac{\partial H}{\partial p_y} \Rightarrow \dot{y} = \frac{2p_x}{m} \Rightarrow \boxed{m\dot{y} = 2p_x}$$

$$\dot{p}_x = -\frac{\partial H}{\partial x} \Rightarrow \boxed{\dot{p}_x = -\frac{1}{2}m\omega^2 y}$$

$$\dot{p}_y = -\frac{\partial H}{\partial y} \Rightarrow \boxed{\dot{p}_y = -\frac{1}{2}m\omega^2 x}$$

Part-C: 2-Mark Numerical Questions

No Question

Part-A: 1-Mark Questions

No Question

Part-B: 3-Mark Questions

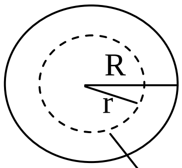
Q1. Consider a sphere of radius R containing a charge with volume density $\rho(r) = 4\pi \in_0 \alpha / r$. The charge is zero outside the sphere. The electromagnetic potentials (ϕ and $\vec{A}$) inside the sphere may be written in many ways. Which of the following values of ϕ and $\vec{A}$ inside the sphere describe the situation correctly?

(a) $\phi = 0,\ \vec{A} = -2\pi\alpha t\hat{r}$ (b) $\phi = 2\pi\alpha r,\ \vec{A} = 0$

(c) $\phi = 0,\ \vec{A} = -\pi\alpha t\hat{r}$ (d) $\phi = \pi\alpha r, \vec{A} = 0$

Ans.: (a)

Solution.:



$$\because \oint_S \vec{E}.d\vec{a} = \frac{Q_{enc}}{\varepsilon_0} \Rightarrow |\vec{E}| \times 4\pi r^2 = \frac{1}{\varepsilon_0}\left(\int_0^r \left(\frac{4\pi \in_0 \alpha}{r}\right) \times 4\pi r^2 dr\right)$$

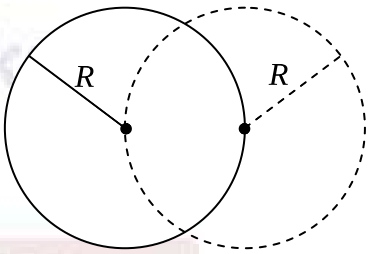
$$\Rightarrow |\vec{E}| \times 4\pi r^2 = \frac{1}{\varepsilon_0} \times 4\pi\varepsilon_0\alpha \times 4\pi\frac{r^2}{2} \quad \Rightarrow \vec{E} = 2\pi\alpha\hat{r}$$

$$\because \vec{E} = -\vec{\nabla}\phi - \frac{\partial\vec{A}}{\partial t}.$$

Let us verify the given option (a) $\phi = 0,\ \vec{A} = -2\pi\alpha t\hat{r}$,

$$\because \vec{E} = -\vec{\nabla}\phi - \frac{\partial\vec{A}}{\partial t} \Rightarrow \vec{E} = -\vec{\nabla}0 - \frac{\partial(-2\pi\alpha t\hat{r})}{\partial t} \quad \Rightarrow \vec{E} = 2\pi\alpha\ \hat{r}$$

Q9. Consider a spherical shell of radius R having a uniform surface charge density σ. Suppose we construct a spherical Gaussian surface having the same radius R but its centre shifted from the charged sphere by a distance R (see the figure). What is the total electric flux $\oint \vec{E} \cdot d\vec{A}$ through the Gaussian surface?



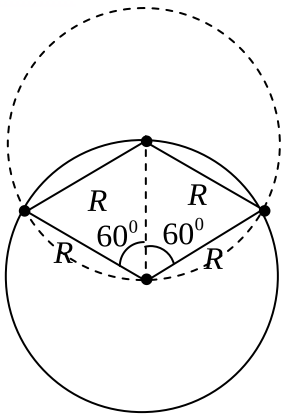
(a) 0 (b) $\pi R^2\sigma$

(c) $2\pi R^2\sigma$ (d) $4\pi R^2\sigma$

Ans.: (b)

Solution.:

Area of the shell under Gaussian Surface is



$$A = \int_0^{60^0}\int_0^{2\pi} R^2 \sin\theta d\theta d\phi = R^2\left[-\cos\theta\right]_0^{60^0} \times 2\pi$$

$$\Rightarrow A = R^2\left[-\cos 60^0 + \cos 0^0\right] \times 2\pi = R^2\left[-\frac{1}{2} + 1\right] \times 2\pi$$

$$\Rightarrow A = \pi R^2$$

$$\because \oint_S \vec{E}.d\vec{a} = \frac{Q_{enc}}{\varepsilon_0} \Rightarrow \oint_S \vec{E}.d\vec{a} = \frac{\sigma \times \pi R^2}{\varepsilon_0}$$

There is some correction in given options. So option (b) should be $\frac{\sigma \pi R^2}{\varepsilon_0}$.

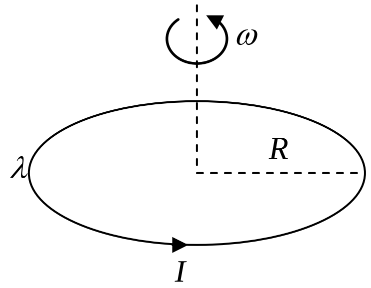
Part-C: 2-Mark Numerical Questions

Q2. A circular ring of radius R with total charge Q_{ring} has uniform linear charge density. It rotates about an axis passing through its centre and perpendicular to its plane with a constant angular speed ω. The magnetic field at the centre is found to be B_0. Another thin circular disk of the same radius R has a constant surface charge density with a total charge Q_{disk}. This disk too rotates about the same axis as the ring with the same constant angular speed ω. The magnetic field at the centre in this case is found to be $10^{-3} B_0$. What is the value of Q_{ring} / Q_{disk} ?

Ans.: **2000**

Solution.:

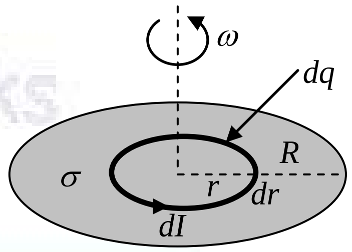
For circular ring, at centre



$$B_0 = \frac{\mu_0 I}{2R} = \frac{\mu_0}{2R} \times \left(\frac{Q_{ring}}{2\pi R} \times R\omega \right) = \frac{\mu_0 Q_{ring} \omega}{4\pi R}$$

$$\because I = \lambda v = \lambda R\omega = \frac{Q_{ring}}{2\pi R} \times R\omega$$

For circular disk, at centre



$$dB = \frac{\mu_0 dI}{2r} = \frac{\mu_0}{2r}\frac{dq}{dt} = \frac{\mu_0}{2r}\frac{\sigma \times 2\pi r dr}{2\pi/\omega} = \frac{\sigma\omega\mu_0 dr}{2}$$

$$\Rightarrow B = \frac{\sigma\omega\mu_0 R}{2} = \frac{Q_{disk}}{\pi R^2}\frac{\omega\mu_0 R}{2} = \frac{\mu_0 Q_{disk}\omega}{2\pi R} = 10^{-3} B_0$$

$$\Rightarrow \frac{\mu_0 Q_{disk}\omega}{2\pi R} = 10^{-3}\frac{\mu_0 Q_{ring}\omega}{4\pi R} \Rightarrow \frac{Q_{ring}}{Q_{disk}} = 2\times 10^3 = 2000$$

Q5. Assume the earth to be an uniform sphere of radius 6400 km and having a uniform electric permittivity of 8.85×10^{-12} Farad/m. What would be the self capacitance (in micro-Farads) of the earth? Round off your answer to the nearest integer.

Ans.: **712**

Solution.:

$$\because C = \frac{Q}{V} = \frac{Q}{Q/4\pi\varepsilon_0 R} = 4\pi\varepsilon_0 R \Rightarrow C = \frac{6400\times 10^3}{9\times 10^9} \approx 712\mu F \qquad \because \frac{1}{4\pi\varepsilon_0} = 9\times 10^9$$

Part-A: 1-Mark Questions

Q3. A particle with energy E is in a bound state of the one-dimensional Hamiltonian $H=-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}+V(x)$. The expectation value of the momentum $\langle p\rangle$

(a) is always zero

(b) depends on the degeneracy of the eigenstate

(c) is zero if and only if the potential symmetric $V(-x)=V(x)$

(d) depends on the energy E of the eigenstate

Ans: (c)

Solution:

The particle in bound state is moving back and forth, and so its average momentum for any quantum state is zero if ψ is real.

Q11. If $\vec{L}$ is the angular momentum operator in quantum mechanics, the value of $\vec{L}\times\vec{L}$ will be

(a) 0 (b) $i\hbar\vec{L}$ (c) $|\vec{L}|$ (d) $\hbar\vec{L}$

Ans: (b)

Solution:

$$\vec{L}\times\vec{L}=\left(L_x\hat{i}+L_y\hat{j}+L_z\hat{k}\right)\times\left(L_x\hat{i}+L_y\hat{j}+L_z\hat{k}\right)$$

$$=\left(L_yL_z-L_zL_y\right)\hat{i}+\left(L_zL_x-L_xL_z\right)\hat{j}+\left(L_xL_y-L_yL_x\right)\hat{k}$$

$$=\left[L_y,L_z\right]\hat{i}+\left[L_z,L_x\right]\hat{j}+\left[L_x,L_y\right]\hat{k}=i\hbar L_x\hat{i}+i\hbar L_y\hat{j}+i\hbar L_z\hat{k}=i\hbar\left(L_x\hat{i}+L_y\hat{j}+L_z\hat{k}\right)=i\hbar\vec{L}$$

Q13. The smallest dimension of the Hilbert space in which we can find operators $\hat{x},\hat{p}$ that satisfy $\left[\hat{x},\hat{p}\right]=i\hbar$ is

(a) 1 (b) 3 (c) 4 (d) ∞

Ans: (d)

Solution:

The commutation relation $\left[x,p\right]=i\hbar$ cannot be satisfied if the dimension of Hilbert space is finite. In finite Hilbert space $\hat{x}$ and $\hat{p}$ can be written as finite matrix term. But for any operator $\hat{A}$ and $\hat{B}$ having finite matrix element

$$trace\left(\left[\hat{A},\hat{B}\right]\right)=0$$

$$trace\left(\hat{A}\hat{B}\right)=trace\left(\hat{B}\hat{A}\right) \qquad \text{Since } \left[\hat{A},\hat{B}\right]=\hat{A}\hat{B}-\hat{B}\hat{A}$$

Therefore, it can't be constant.

Q18. Positronium is a short lived bound state of an electron and a positron. The energy difference between the first excited state and ground state of positronium is expected to be around

(a) four times that of the Hydrogen atom

(b) twice that of the Hydrogen atom

(c) half that of the Hydrogen atom

(d) the same as that of the Hydrogen atom

Ans: (c)

Solution:

For Positronium: $E_n = \frac{-13.6}{2n^2}(\text{eV}) = -\frac{6.8}{n^2}(\text{eV})$

$$\Delta E_p = E_{n=2} - E_{n=1} = -\frac{6.8}{2^2} + \frac{6.8}{1^2} = -1.7 + 6.8 \quad \Rightarrow \Delta E_p = 5.1\text{eV}$$

For H-atom: $E_n = -\frac{13.6}{n^2}(\text{eV})$

$$\Delta E_H = E_{n=2} - E_{n=1} = -\frac{13.6}{2^2} + \frac{13.6}{12} = 10.2\,\text{eV} \Rightarrow \Delta E_H = 10.2\,\text{eV}$$

Thus $\boxed{\Delta E_p = \frac{\Delta E_H}{2}}$

Q20. A one-dimensional box contains three identical particles in the ground state of the system. Find the ratio of total energies of these particles if they were spin-$\frac{1}{2}$ fermions, to that if they were bosons.

(a) 1 (b) $\frac{14}{3}$ (c) 2 (d) $\frac{1}{3}$

Ans: (c)

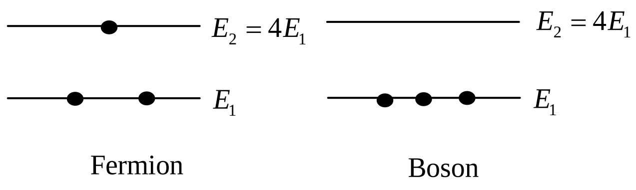
Solution:

In case of spin- $\frac{1}{2}$ fermions;

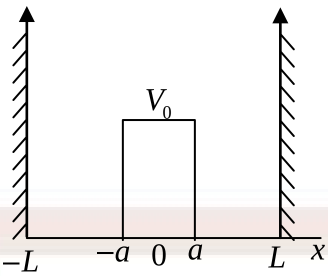
$$E_F = 2E_1 + E_2 = 2E_1 + 4E_1 = 6E_1$$

In case of Bosons; $E_B = 3E_1$

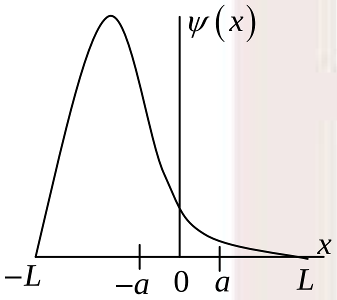
Thus $\frac{E_F}{E_B} = \frac{6E_1}{3E_1} = 2$



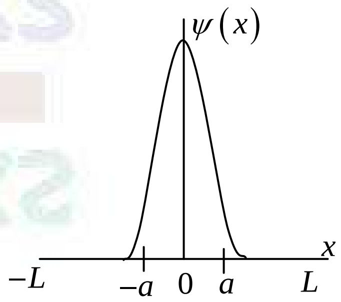
Q23. A quantum particle is moving in one dimension between rigid walls at $x=-L$ and $x=L$, under the influence of a potential (see figure). The potential has the uniform value V_0 between $-a<x<a$, and is 0 otherwise. Which one of the following graphs qualitatively represent the ground state wavefunction of this system? (You can assume that $a \ll L,\ V_0 \gg \pi^2/8mL^2$).



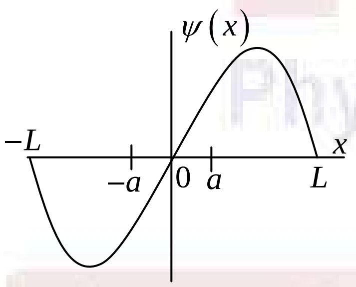
(a)



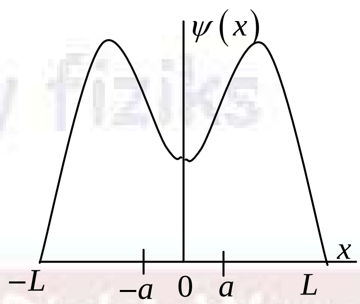
(b)



(c)



(d)



Ans: (d)

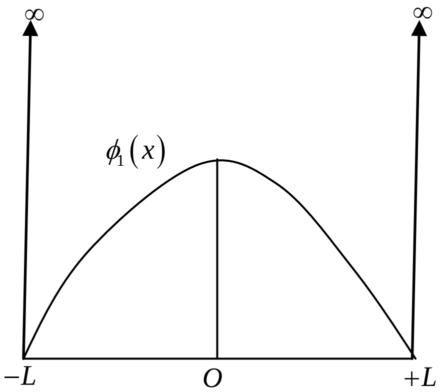
Solution:

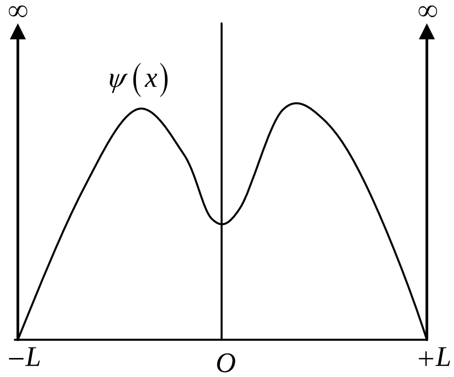
The ground state of wave function including first order correction is

$$\psi(x)=\phi_1(x)+\psi_1'(x) \quad \text{where } \phi_1(x)=\sqrt{\frac{2}{L}}\cos\left(\frac{\pi x}{L}\right) \text{ and } \psi_1'(x)=\sum_{m\neq 1}\frac{\langle \phi_m|H'|\phi_1\rangle}{E_1^{(0)}-E_m^{(2)}}|\phi_m\rangle$$

Since $E_m^{(2)} > E_1^{(c)}$, thus $\psi_1'(x)$ is negative.

The ground state wave function will have cusp at the centre. The correct answer is (d)





Part-B: 3-Mark Questions

Q5. Consider the normalized wave function $\psi = a\psi_0 + b\psi_1$ for a one-dimensional simple harmonic oscillator at some time, where ψ_0 and ψ_1 are the normalized ground state and the first excited state respectively, and a, b are real numbers. For what values of a and b, the magnitude of expectation value of x, i.e. $|\langle x \rangle|$, is maximum?

(a) $a = -b = 1/\sqrt{2}$ (b) $a = b = 1/\sqrt{2}$

(c) $a = 1,\ b = 0$ (d) $a = 0,\ b = 1$

Ans: (b)

Solution:

$|\psi\rangle = a|\psi_0\rangle + b|\psi_1\rangle = a|0\rangle + b|1\rangle$

$\because \langle\psi|\psi\rangle = 1 \Rightarrow |a|^2 + |b|^2 = 1 \Rightarrow a^2 + b^2 = 1$

Now $\langle\hat{x}\rangle = \langle\psi|\hat{x}|\psi\rangle = a^2\langle 0|\hat{x}|0\rangle + b^2\langle 1|\hat{x}|1\rangle + ab\langle 0|\hat{x}|1\rangle + ba\langle 1|\hat{x}|0\rangle$

Since $\langle 0|\hat{x}|0\rangle = \langle 1|\hat{x}|1\rangle = 0 \qquad \Rightarrow \langle\hat{x}\rangle = ab\langle 0|\hat{x}|1\rangle + ab\langle 1|\hat{x}|0\rangle$

where $\langle 0|\hat{x}|1\rangle = \sqrt{\frac{\hbar}{2m\omega}}\left(\langle 0|\hat{a} + \hat{a}^+|1\rangle\right) = \sqrt{\frac{\hbar}{2m\omega}}\left(\langle 0|\hat{a}|1\rangle + \langle 0|\hat{a}^+|1\rangle\right] = \sqrt{\frac{\hbar}{2m\omega}}$

since $\langle 0|a|1\rangle = 1$ and $\langle 0|a^+|1\rangle = 0$

Similarly $\langle 1|\hat{x}|0\rangle = \sqrt{\frac{\hbar}{2m\omega}}$

$\therefore\ \langle\hat{x}\rangle = 2ab\sqrt{\frac{\hbar}{2m\omega}} = 2a\sqrt{1-a^2}\sqrt{\frac{\hbar}{2m\omega}} \qquad \because b = \pm\sqrt{1-a^2}$

$$\frac{\partial\langle\hat{x}\rangle}{\partial a} = 0 \Rightarrow \sqrt{1-a^2} + \frac{a}{2\sqrt{1-a^2}}(-2a) = 0 \Rightarrow \frac{1-a^2-a^2}{\sqrt{1-a^2}} = 0 \quad \Rightarrow \boxed{a = \pm\frac{1}{\sqrt{2}}}$$

Since $b = \pm\sqrt{1-a^2} \Rightarrow \boxed{b = \pm\frac{1}{\sqrt{2}}}$

For maxima; $\frac{\partial^2 \langle x \rangle}{\partial a^2} < 0 \Rightarrow a = b = \frac{1}{\sqrt{2}}$; For minima $\frac{\partial^2 \langle x \rangle}{\partial a^2} < 0 \Rightarrow a = -b = \frac{1}{\sqrt{2}}$

Thus correct answer is (b)

Q8. Consider a 4-dimensional vector space V that is a direct product of two 2-dimensional vector spaces V_1 and V_2. A linear transformation H acting on V is specified by the direct product of linear transformations H_1 and H_2 acting on V_1 and V_2, respectively. In a particular basis,

$$H_1 = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}, \quad H_2 = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

What is the lowest eigenvalue of H?

(a) 1 (b) $\frac{3}{2}$ (c) $3-\sqrt{5}$ (d) $\frac{1}{2}\left(3-\sqrt{5}\right)$

Ans: (c)

Solution:

$V = V_1 \times V_2$; $HV = H_1V_1 \times H_2V_2$

$H_1 = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}$ and $H_2 = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$

Eigenvalues of $H_1 = 3, 2$;

Eigenvalues of H_2; $[H_2 - \lambda I] = 0 \Rightarrow \begin{bmatrix} 2-\lambda & 1 \\ 1 & 1-\lambda \end{bmatrix} = 0$

$$\Rightarrow (2-\lambda)(1-\lambda) - 1 = 0 \Rightarrow \lambda^2 - 3\lambda + 1 = 0 \Rightarrow \lambda = \frac{3 \pm \sqrt{9-4}}{2} = \frac{3 \pm \sqrt{5}}{2}$$

$\therefore \lambda_1 = \frac{3+\sqrt{5}}{2}$ and $\lambda_2 = \frac{3-\sqrt{5}}{2}$

The smallest eigenvalues is $\lambda = \frac{3-\sqrt{5}}{2}$

Q13. A particle is in the nth energy eigenstate of an infinite one-dimensional potential well between $x = 0$ and $x = L$. Let P be the probability of finding the particle between $x = 0$ and $x = 1/3$. In the limit $n \to \infty$, the value of P is

(a) $1/9$ (b) $2/3$ (c) $1/3$ (d) $1/\sqrt{3}$

Ans: (c)

Solution:

$$\phi_n(x)=\sqrt{\frac{2}{L}}\sin\left(\frac{2\pi}{L}x\right)$$

$$P=\int_0^{L/3}|\phi_n(x)|^2\,dx=\frac{2}{L}\int_0^{L/3}\sin^2\left(\frac{n\pi}{L}x\right)dx$$

$$=\frac{1}{L}\int_0^{L/3}\left(1-\cos\left(\frac{2n\pi}{L}x\right)\right)dx=\frac{1}{L}\left[\int_0^{L/3}dx-\int_0^{L/3}\cos\left(\frac{2n\pi}{L}x\right)dx\right]$$

$$\Rightarrow P=\frac{1}{L}\left[\frac{1}{3}-\frac{\sin\left(\frac{2n\pi}{L}x\right)}{\frac{2n\pi}{L}}\Bigg|_0^{L/3}\right]=\frac{1}{L}\left[\frac{L}{3}-\frac{L}{2n\pi}\left(\sin\left(\frac{2n\pi}{3}\right)\right)\right]=\frac{1}{3}-\frac{\sin\left(\frac{2n\pi}{3}\right)}{2n\pi}$$

when $n\to\infty\Rightarrow\boxed{P=\frac{1}{3}}$

Part-C: 2-Mark Numerical Questions

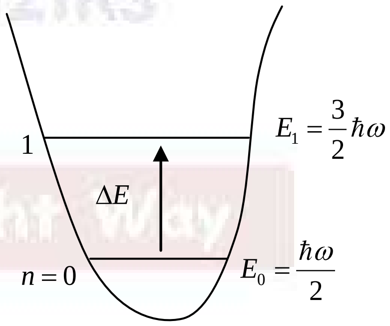
Q4. The uncertainty Δx in the position of a particle with mass m in the ground state of a harmonic oscillator is $2\hbar/mc$. What is the energy (in units of $10^{-4}mc^2$) required to excite the system to the first excited state?

Ans: 1250

Solution:

$\Delta E=\hbar\omega$; Now $\Delta x=\frac{2\hbar}{mc}$

Since



$$\Delta x=\sqrt{\langle x^2\rangle-\langle x\rangle^2}=\sqrt{\langle x^2\rangle}=\sqrt{\frac{\hbar}{2m\omega}(2n+1)}=\sqrt{\frac{\hbar}{2m\omega}}$$

Thus $\sqrt{\frac{\hbar}{2m\omega}}=\frac{2\hbar}{mc}\Rightarrow\frac{\hbar}{2m\omega}=\frac{4\hbar^2}{m^2c^2}$

$$\Rightarrow\hbar\omega=\frac{1}{8}mc^2=\frac{mc^2}{8}\times10^{-4}\times10^{4}$$

$$\therefore\ \Delta E=\hbar\omega=\frac{10000}{8}\left(10^{-4}\text{mc}^2\right)=1250\left(10^{-4}\text{mc}^2\right)$$

Part-A: 1-Mark Questions

Q8. An ideal gas at temperature T is composed of particles of mass m, with the x-component of velocity v_x. The average value of $|v_x|$ is

(a) 0 (b) $\sqrt{3k_BT/m}$ (c) $\sqrt{k_BT/2\pi m}$ (d) $\sqrt{2k_BT/\pi m}$

Ans: (d)

Solution.:

$$\langle |u_x| \rangle = \frac{2\int_0^\infty |u_x| P(u_x)}{\int_{-\infty}^{\infty} P(u_x)} = \frac{I_1}{I_2} \quad \text{where } I_2 = 1.$$

$$P(u_x) = \frac{dN_{u_x}}{N} = \left[\frac{m}{2\pi K_BT}\right]^{1/2} e^{-\frac{mu_x^2}{2K_BT}} du_x$$

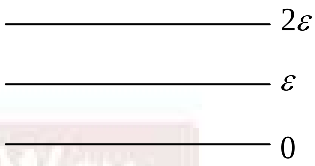
$$I_1 = 2\left[\frac{m}{2\pi K_BT}\right]^{1/2} \int_0^\infty |u_x| e^{-\frac{mu_x^2}{2K_BT}} du_x = 2\left[\frac{m}{2\pi K_BT}\right]^{1/2} \frac{1}{2}\frac{2K_BT}{m} = \sqrt{\frac{2K_BT}{\pi m}}$$

Q14. Consider a system consisting of three non-degenerate energy levels, with energies 0, $\in$ and $2\in$. In the limit of infinite temperature $T \to \infty$, the probability of finding a particle in the ground state is

(a) 0 (b) $1/2$ (c) $1/3$ (d) 1

Ans: (c)

Solution.:



$$Z = 1 + e^{-\beta\varepsilon} + e^{-2\beta\varepsilon}$$

$$P(\varepsilon = 0) = \frac{e^{-\beta \cdot 0}}{Z} = \frac{1}{1 + e^{-\frac{\varepsilon}{KT}} + e^{\frac{-2\varepsilon}{KT}}}$$

In the limit of $T \to \infty$; $P(\varepsilon = 0) = \frac{1}{1+1+1} = \frac{1}{3}$

Q25. The free energy density of a gas at a constant temperature is given by $f(\rho) = C\rho \ln(\rho/\rho_0)$, where ρ represents the density of the gas, while C and ρ_0 are positive constants. The pressure of the system is

(a) $C\rho$ (b) $C\rho^2/\rho_0$ (c) $C\rho_0 \ln(\rho/\rho_0)$ (d) $C\rho \ln(\rho/\rho_0)$

Ans.: (a)

Solution.:

$f(\rho)=C\rho\ln(\rho/\rho_0)$, ρ in density of gas C and ρ_0 are positive constants.

$dF=-SdT-PdV$; At fix T, $P=-\left(\frac{\partial F}{\partial V}\right)$

$$F=Vf(\rho)=VC\rho\ln(\rho/\rho_0)=CM\ln(M/V\rho_0) \qquad \because \rho=\frac{M}{V}$$

$$P=-\left(\frac{\partial F}{\partial V}\right)_T=-CM\frac{\rho_0 V}{M}\frac{M}{\rho_0}\left\{-\frac{1}{V^2}\right\}=CM/V=C\rho$$

Part-B: 3-Mark Questions

Q6. M grams of water at temperature T_a is adiabatically mixed with an equal mass of water at temperature T_b, keeping the pressure constant. Find the change in entropy of the system (specific heat of water is C_p).

(a) $\Delta S=MC_p\ln\left[1-\frac{(T_a-T_b)^2}{4T_aT_b}\right]$ (b) $\Delta S=MC_p\ln\left[1+\frac{(T_a+T_b)^2}{4T_aT_b}\right]$

(c) $\Delta S=MC_p\ln\left[1+\frac{(T_a-T_b)^2}{4T_aT_b}\right]$ (d) $\Delta S=MC_p\ln\left[\frac{T_a+T_b}{(4T_aT_b)^{1/2}}\right]$

Ans.: (c)

Solution.:

$M,T_a \leftrightarrow M,T_b$ Let $T_a>T_b$

$$MC_p(T_a-T_f)=MC_p(T_f-T_b)\Rightarrow T_f=\frac{T_a+T_b}{2}$$

$$\Delta S_1=MC_p\ln\left(\frac{T_f}{T_a}\right)=MC_p\ln\frac{T_a+T_b}{2T_a}; \qquad \Delta S_2=MC_p\ln\left(\frac{T_f}{T_b}\right)=MC_p\ln\frac{T_a+T_b}{2T_b}$$

$$\Delta S=\Delta S_1+\Delta S_2=MC_p\ln\frac{\{(T_a+T_b)/2\}^2}{T_aT_b}=MC_p\ln\frac{(T_a+T_b)^2}{4T_aT_b}$$

$$=MC_p\ln\left[\frac{4T_aT_b+T_a^2+T_b^2-2T_aT_b}{4T_aT_b}\right]$$

$$\Delta S=MC_p\ln\left[1+\frac{(T_a-T_b)^2}{4T_aT_b}\right]$$

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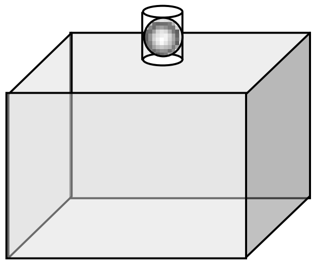
Q10. A large box, of volume V is fitted with a vertical glass tube of cross-sectional area A, in which a metal ball of mass m fits exactly. The box contains an ideal gas at a pressure slightly higher than atmospheric pressure P because of the weight of the ball. If the ball is displaced slightly from equilibrium, find the angular frequency ω of simple harmonic oscillations. Assume adiabatic behaviour, with ratio of specific heats $\gamma = C_P / C_V$.

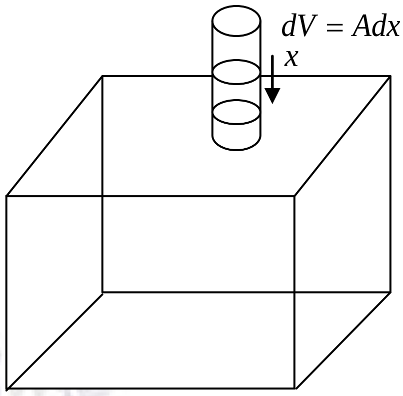


(a) $\omega = \sqrt{\dfrac{A^2\left(P+mg/A\right)}{2\gamma Vm}}$ (b) $\omega = \sqrt{\dfrac{2\gamma A^2\left(P+mg/A\right)}{Vm}}$

(c) $\omega = \sqrt{\dfrac{A^2\left(P+mg/A\right)}{\gamma Vm}}$ (d) $\omega = \sqrt{\dfrac{\gamma A^2\left(P+mg/A\right)}{Vm}}$

Ans: (d)

Solution.:



Let initial pressure be P_1; $P_1 = P + \dfrac{mg}{A}$ and $V_1 = V$

Final pressure when ball is displaced slightly above

$P_2 = P_1 + \Delta P$ Pressure$\uparrow$, dx

$V_2 = V - \Delta V,\ \Delta V = Adx$

Applying eqation

$$P_1V_1^\gamma = P_2V_2^\gamma = \left(P_1+\Delta P\right)\left(V-\Delta V\right)^\gamma$$

$$= P_1V^\gamma\left[1+\frac{\Delta P}{P_1}\right]\left[1-\frac{\Delta V}{V}\right]^\gamma$$

$$1 = \left[1+\frac{\Delta P}{P_1}\right]\left[1-\gamma\frac{\Delta V}{V}\right] = 1-\gamma\frac{\Delta V}{V}+\frac{\Delta P}{P_1}-\gamma\frac{\Delta P}{P_1}\frac{\Delta V}{V} \Rightarrow \Delta P = \gamma P_1\frac{\Delta V}{V}$$

Restoring force on ball is $F = -A\Delta P = \dfrac{-\gamma AP_1}{V}Ax = \dfrac{-\gamma A^2P_1x}{V}$

Acceleration of the ball is $a = \dfrac{f}{m} = -\dfrac{\gamma A^2P_1}{mV}x \Rightarrow a \propto x \quad \therefore\ SHM$

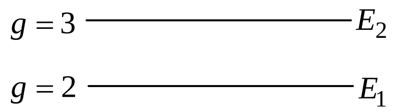
$$\therefore \omega = \sqrt{\frac{\gamma A^2}{mV}P_1} \Rightarrow \omega = \sqrt{\frac{\gamma A^2}{mV}\left(p+\frac{mg}{A}\right)}$$

Part-C: 2-Mark Numerical Questions

Q3. Five distinguishable particles are distributed in energy levels E_1 and E_2 with degeneracy of 2 and 3 respectively. Find the number of microstates with three particles in energy level E_1 and two particles in E_2.

Ans: 0720

Solution.: No of distinguishable particles $=5$



Out of these $5, 3$ have to be in E_1 and 2 in E_2.

$\therefore$ Possible arrangements are $\Omega_1 = \dfrac{5!}{3!2!} = \dfrac{5\cdot 4\cdot 3!}{3!\cdot 2} = 10$

Considering degeneracy, Level E_1 is 2 fold degenerate

No of ways particle 1 can be placed $=2$

No of ways particle 2 can be placed $=2$

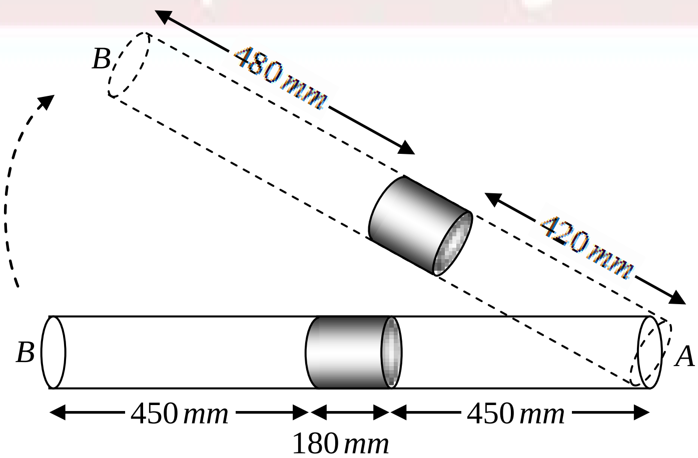
No of ways particle 3 can be placed $=2$

$\therefore$ No of ways of arranging 3 particles in E_1 with $g=2$ are $=2^3$

Similarly no of ways of arranging 2 particles in E_2 with $g=3$ are $=3^2$

$\therefore$ Total microstates for arranging 5 particles $\Omega_{Total} = \dfrac{5!}{3!2!}2^3 3^2 = 720$

Q8. A thin tube of length $1080\,mm$ and uniform cross-section is sealed at both ends, and placed horizontally on a table. At the exact center of the tube is a mercury (Hg) pellet of length $180\,mm$. The pressure of the air on both sides of the mercury pellet is P_0. When the tube is held at an angle of 60 degrees with the vertical, the length of the air column above and below the Hg become $480\,mm$ and $420\,mm$, respectively. Assuming the temperature of the system to be constant, calculate the pressure P_0 in mm of Hg.



Ans: 0671

Solution.:

When tube is held at an angle of 60° with vertical

Length of upper part $=\ell_1 = 48\,cm$; Length of Lower part $=\ell_2 = 42\,cm$

When tube lies horizontal, length of air column on both sides

$$=\ell_0 = \frac{\ell_1+\ell_2}{2} = \frac{48+42}{2} = 45\,cm$$

Let P_1 and P_2 be the pressures in upper and lower parts when tube is kept inclined. As T is uniform throught, we apply Boyle's law $PV = cons$.

For upper part $P_1\ell_1 A = P_0\ell_0 A \;\Rightarrow P_1 = \dfrac{P_0\ell_0}{\ell_1}$ (1)

Similarly for the lower part, $P_2\ell_2 A = P_0\ell_0 A \Rightarrow P_2 = \dfrac{P_0\ell_0}{\ell_2}$ (2)

When the tube is held vertically at 60° the Hg column will be displaced to lower end and

$$P_2 = P_1 + \frac{mg}{A}\cos 60^\circ \Rightarrow \frac{P_0\ell_0}{\ell_2} = \frac{P_0\ell_0}{\ell_1} + \frac{mg}{2A} \Rightarrow P_0\ell_0\left[\frac{1}{\ell_2}-\frac{1}{\ell_1}\right] = \frac{mg}{2A}$$

$$\Rightarrow P_0 = \frac{mg}{2A\ell_0\left(\dfrac{1}{\ell_2}-\dfrac{1}{\ell_1}\right)}$$

The pressure P_0 is equal to height h of mercury; $P_0 = \rho g h$, also $m = (18\,cm)A\rho$

$$\therefore \rho g h = \frac{18A\rho g}{2A\ell_0\left(\dfrac{1}{\ell_2}-\dfrac{1}{\ell_1}\right)} \Rightarrow h = \frac{18}{2\ell_0\left[\dfrac{1}{\ell_2}-\dfrac{1}{\ell_1}\right]} = \frac{18cm}{2\times 45\,cm\left[\dfrac{1}{42}-\dfrac{1}{48}\right]} \Rightarrow h = 67.11cm$$

$$\Rightarrow P_0 = 67.11cm = 671.1mm \text{ of } Hg$$

Part-A: 1-Mark Questions

Q1. A negative logic is the one in which the 0's and the 1's in the truth tables are interchanged. In such a negative logic, the normal NAND gate would behave like a

(a) NOR gate (b) AND gate

(c) OR gate (d) NAND gate

Ans.: (a)

Solution.:

$+ve$ Logic	$-ve$ Logic
$0 \rightarrow$ Low	$1 \rightarrow$ Low
$1 \rightarrow$ High	$0 \rightarrow$ High

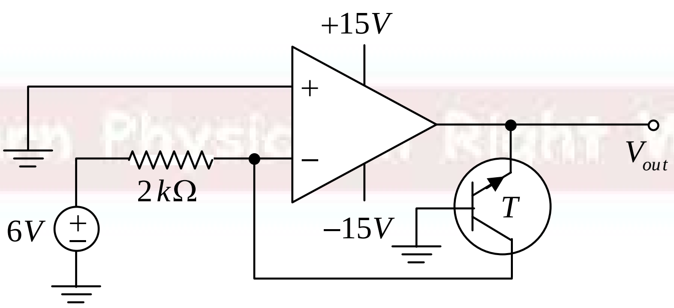
The NAND Gate Truth Table

Inputs		Outputs
A	B	X
0	0	1
0	1	1
1	0	1
1	1	0

Inputs		Outputs
A	B	X
1	1	0
1	0	0
0	1	0
0	0	1

NOR Gate

Q5. An ideal op-amp and a silicon transistor T are used in the following circuit. Find the output voltage V_{out}



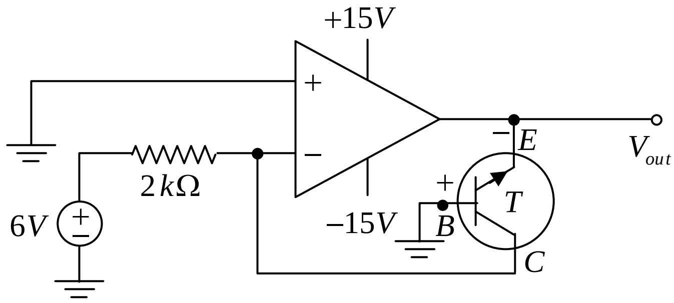
(a) $+5.3\,V$ (b) $-0.7\,V$ (c) $+0.7V$ (d) $-15V$

Ans.: (b)

Solution.:

$\because V_{BE} = 0.7\,\text{Volts}$

$V_{out} = V_{EB} = -V_{BE} = -0.7\,\text{Volts}$



Q12. In an open circuited p-n junction diode, the barrier voltage at the junction is generated due to

(a) Minority carriers in the p and n sides

(b) Majority carriers in the p and n sides

(c) Immobile negative charge in the p-side and positive charge in the n-side

(d) Immobile positive charge in the p-side and negative charge in the n-side

Ans.: (c)

Solution.:

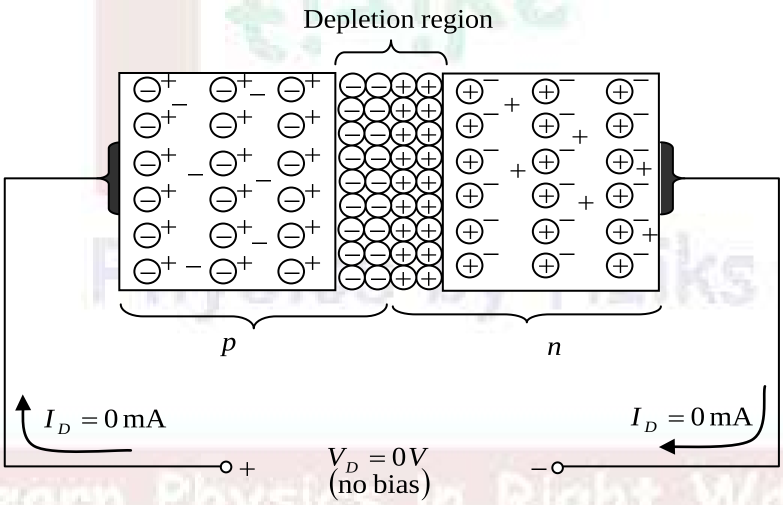
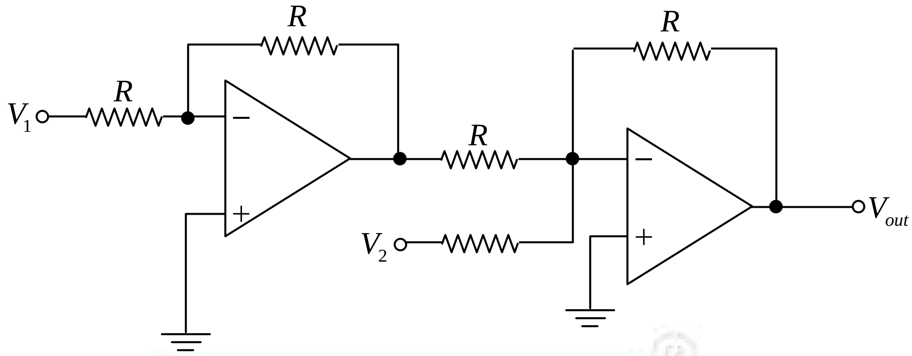


Figure: *p-n* junction with no applied bias.

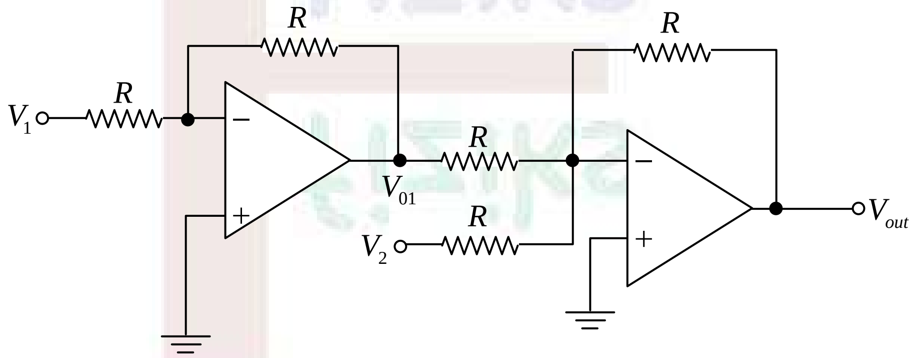
Q15. In the figure below with ideal op-amps, the value of $R = 10\,k\Omega$, $V_1 = -10\,mV$, and $V_2 = -30\,mV$. Calculate V_{out}.



(a) $+40\,mV$ (b) $-40\,mV$ (c) $+20\,mV$ (d) $-20\,mV$

Ans.: (c)

Solution.:



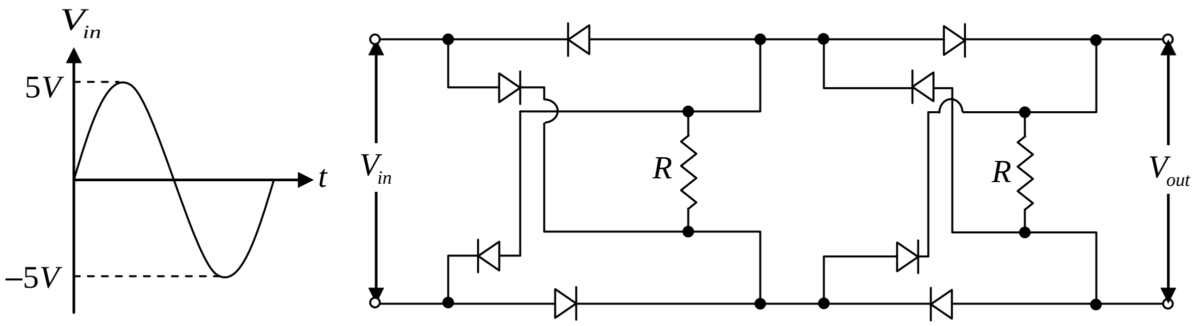
$$V_{01} = -\frac{R}{R}V_1 = -V_1$$

$$V_{out} = V_{01}' + V_2' = -\frac{R}{R}V_{01} - \frac{R}{R}V_2 = -(-V_1) - V_2 = V_1 - V_2$$

$$\Rightarrow V_{out} = V_1 - V_2 = -10\,mV - (-30\,mV) = +20\,mV$$

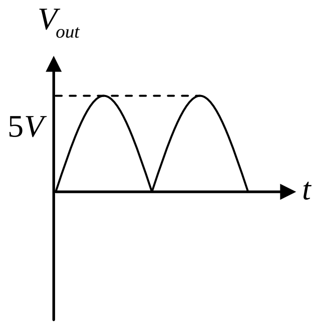
Part-B: 3-Mark Questions

Q7. The circuit given in the figure below is composed of ideal diodes and resistances R. The input waveform is shown on the left.

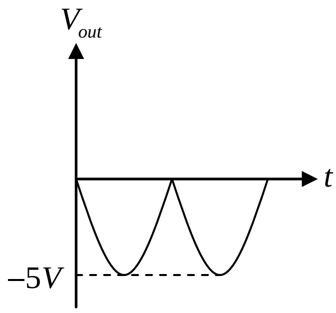


The output waveform would be

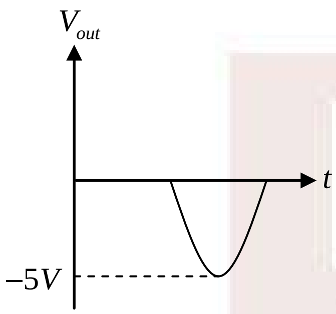
(a)



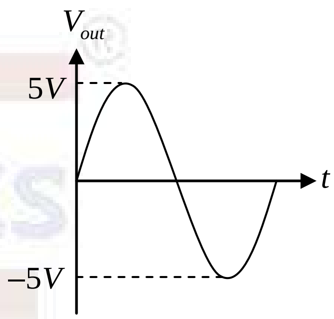
(b)



(c)



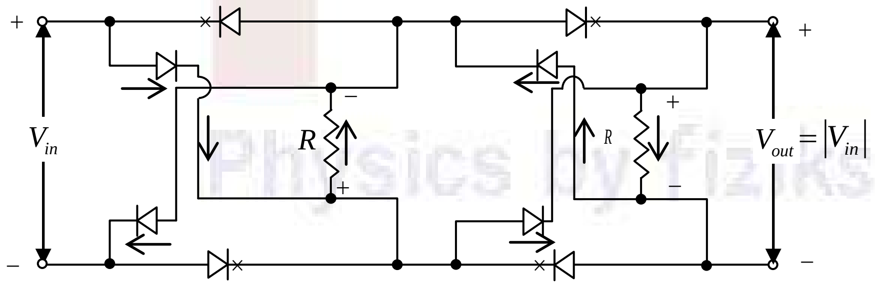
(d)



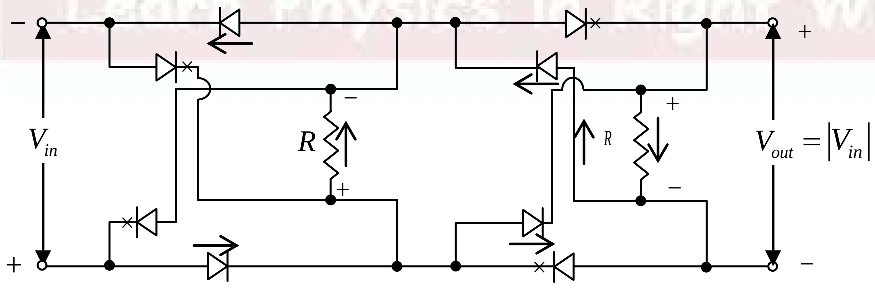
Ans.: (a)

Solution.:

Positive half Cycle: Identify which diode is ON and which diode is OFF.



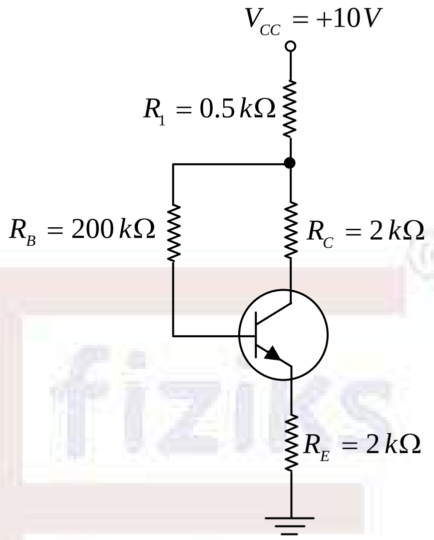
Negative half Cycle: Identify which diode is ON and which diode is OFF.



So in both half cycles output will be positive. So option (a) is correct.

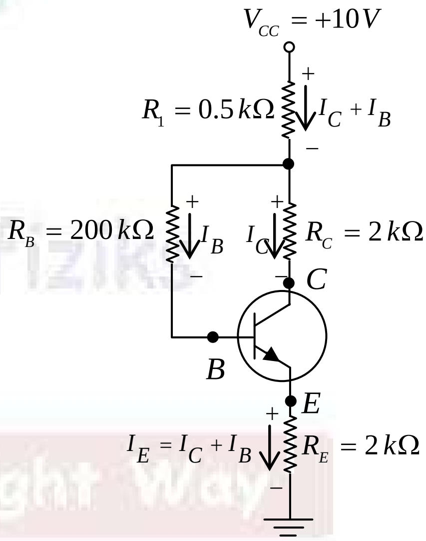
Part-C: 2-Mark Numerical Questions

Q9. In the following transistor circuit $R_1 = 0.5\,k\Omega,\ R_E = R_C = 2k\Omega, R_B = 200\,k\Omega$, $\beta = \frac{I_C}{I_B} = 100$, $V_{CC} = 10V, V_{BE} = 0.7V$. Determine the V_{CE} in mV.



Ans.: 950

Solution.:



Apply KVL in input section:

$$-10V + 0.5k\left(I_C + I_B\right) + 200k \times I_B + 0.7V + 2k \times I_E = 0$$

$$-10V + 0.5k\left(\beta+1\right)I_B + 200k \times I_B + 0.7V + 2k \times \left(\beta+1\right)I_B = 0$$

$$-10V + 0.5k \times 101 I_B + 200k \times I_B + 0.7V + 2k \times 101 I_B = 0$$

$$\Rightarrow I_B = \frac{9.3V}{0.5k \times 101 + 200k + 2k \times 101} = \frac{9.3V}{\left(50.5 + 200 + 202\right)k}$$

$$\Rightarrow I_B = \frac{9.3V}{452.5k} = 0.02mA$$

Apply KVL in output section:

$$-10V + 0.5k\left(\beta+1\right)I_B + 2k \times \beta I_B + V_{CE} + 2k \times \left(\beta+1\right)I_B = 0$$

$$\Rightarrow V_{CE} = 10V - \left[0.5k \times 101 + 2k \times 100 + +2k \times 101\right] \times 0.02mA$$

$$\Rightarrow V_{CE} = 10V - \left[50.5 + 200 + +202\right] \times 0.02\ V$$

$$\Rightarrow V_{CE} = 10V - \left(452.5 \times 0.02\right)V = \left(10 - 9.05\right)V = 0.95V = 950mV$$

Part-A: 1-Mark Questions

Q4. A glass of radius R and refractive index n acts like a lens with focal length

(a) $-\frac{nR}{2(n-1)}$ (b) $+\frac{nR}{2(n-1)}$

(c) $-\frac{nR}{2(n-1)^2}$ (d) $+\frac{nR}{2(n-1)^2}$

Ans: (b)

Solution:

For thick lens; $\frac{1}{f}=(n-1)\left[\frac{1}{R_1}-\frac{1}{R_2}+\frac{(n-1)t}{nR_1R_2}\right]$

$R_1=+R$, $R_2=-R$, $t=2R$

$$\frac{1}{f}=(n-1)\left[\frac{1}{R}-\frac{1}{-R}+\frac{(n-1)2\cancel{R}}{n(+\cancel{R})(-R)}\right]\Rightarrow\frac{1}{f}=\frac{2(n-1)}{R}\left[1-\frac{n-1}{n}\right]\quad\Rightarrow\boxed{f=\frac{nR}{2(n-1)}}$$

Q16. A flat soap film has a uniform thickness of 510 nm. White light (having wavelengths in the range of about 390 – 700 nm) is incident normally on the film. If the refractive index of the soap is 1.33, what will be the dominant colour of the reflected light?

(a)Violet (b) Green (c) Red (d) White

Ans.: (b)

Solution:

Condition of maxima in reflected light; $2\mu t=(2n+1)\frac{\lambda}{2}$; $n=0,1,2,....$

$$\lambda=\frac{4\mu t}{(2n+1)}=\frac{4\times1.33\times510}{(2n+1)}\text{nm}\Rightarrow\lambda=\frac{2713}{(2n+1)}\text{nm}$$

On substituting $n=0,1,2,3,4....$ we get

$\lambda=2713\,\text{nm},904.3\,\text{nm},542.6\,\text{nm},387.5\,\text{nm},301.4\,\text{nm}$ etc.

Out of these values $\lambda=542.6\,\text{nm}$ lies in white light wavelength range $(390-700\,\text{nm})$, so the Dominant colour will be corresponding to $\lambda=542.6\,\text{nm}$

Violet $\rightarrow$390 nm; Red $\rightarrow$ 700 nm; so, $\lambda=542.6$ nm will be corresponding to green colour.

Q21. A monochromatic linearly polarized light with electromagnetic field $\vec{E} = E_0 \sin(\omega t - kz)(\hat{x} + \hat{y})$ is incident normally on a birefringent calcite crystal. The wavelength of the wave is $590\,nm$ and the refractive indices of the crystal along the x- directions and y- directions are 1.66 and 1.49, respectively. If the thickness of the crystal is $434\,nm$, what will be the polarization of the light that emerges from the crystal?

(a) Linearly polarized along the same axis as the incident light

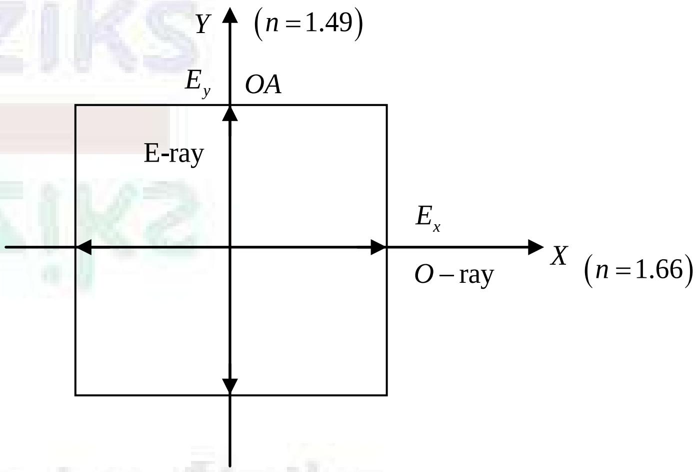
(b) Linearly polarized but along a different axis than the incident light

(c) Circularly polarized

(d) Neither linearly nor circularly polarized but elliptically polarized

Ans: (d)

Solution:



Incident light is linearly polarized light, given by

$E_x = E_0 \sin(\omega t - kz)$;

$E_y = E_0 \sin(\omega t - kz)$

Calcite crystal is a negative crystal $(n_E < n_0)$.

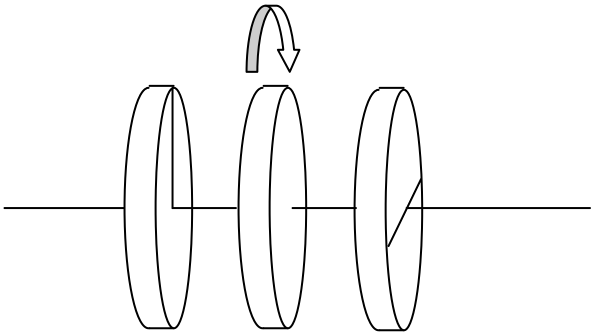
Phase difference introduced between O-ray and E-ray by the crystal $\delta = \frac{2\pi}{\lambda}(n_0 - n_E)t = \frac{2\pi}{590}(1.66 - 1.49) \times 434$

$\delta = 0.25\pi = \frac{\pi}{4}$ radian.

Emergent light will be given by $E_x = E_0 \sin \omega t$; $E_y = E \sin\left(\omega t + \frac{\pi}{4}\right)$ which is elliptically polarized light.

Part-B: 3-Mark Questions

Q3. An ideal polariser is placed in between two crossed polarisers in a coaxial geometry as shown. The middle polariser is rotated at the angular speed of ω about the common axis. If unpolarized light of intensity I_0 is incident on this system, the emergent intensity of the light would be



(a) $\frac{I_0}{8}[1-\cos 4\omega t]$ (b) $\frac{I_0}{16}[1-\cos 4\omega t]$

(c) $\frac{I_0}{16}[1-\cos \omega t]$ (d) $\frac{I_0}{16}\left[1-\frac{1}{2}\cos \omega t\right]$

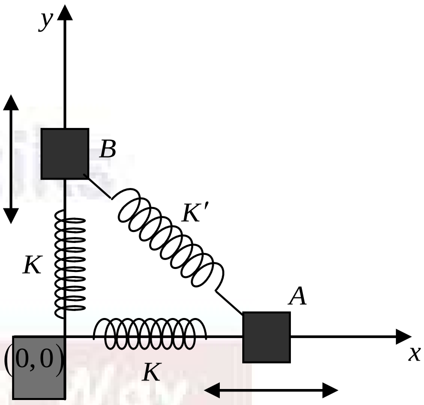
Ans: (b)

Solution: Let pass axis of middle polarize makes an θ with the pass axis of first polariser.

$$I_1=\frac{I_0}{2},\ I_2=\frac{I_0}{2}\cos^2\theta,\ I_3=\frac{I_0}{2}\cos^2\theta\cos^2\left(\frac{\pi}{2}-\theta\right)=\frac{I_0}{8}\sin^2 2\theta$$

$$I_3=\frac{I_0}{16}(1-\cos 4\theta)=\frac{I_0}{16}(1-\cos 4\omega t)$$

Q14. Two equal masses A and B are connected to a fixed support at the origin by two identical springs with spring constant K and the same unstretched length L. They are also connected to each other by a spring with spring constant K' and unstretched length $\sqrt{2}L$. The equilibrium position, with all springs unstretched, is shown in the figure. If A is constrained to move only along the x axis and B is constrained to move only along the y axis, then the angular frequencies $\omega_1,\ \omega_2$ of the normal modes are

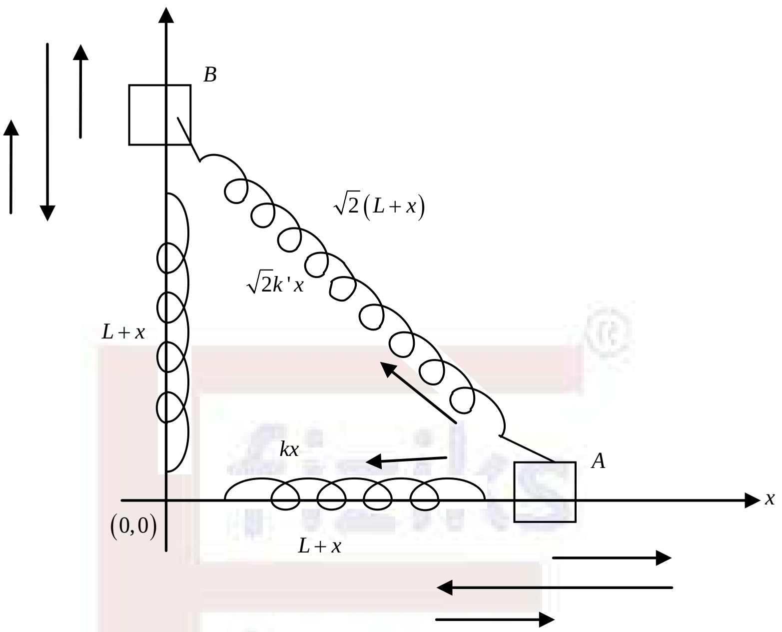


(a) $\omega_1=\sqrt{\frac{K}{m}},\ \omega_2=\sqrt{\frac{K+K'}{m}}$ (b) $\omega_1=\sqrt{\frac{K}{m}},\ \omega_2=\sqrt{\frac{2K'}{m}}$

(c) $\omega_1=\sqrt{\frac{2K}{m}},\ \omega_2=\sqrt{\frac{K+K'}{m}}$ (d) $\omega_1=\sqrt{\frac{K}{m}},\ \omega_2=\sqrt{\frac{K+2K'}{m}}$

Ans: (a)

Solution:



In case of normal mode both blocks oscillate with the same frequencies. There are two ways in which A and B can oscillate with same frequencies.

(1) Total force acting on block A

$$F=-kx-\sqrt{2}k'x\cos 45^\circ$$

$$ma=-(K+K')x \Rightarrow a=-\frac{K+K'}{m}x$$

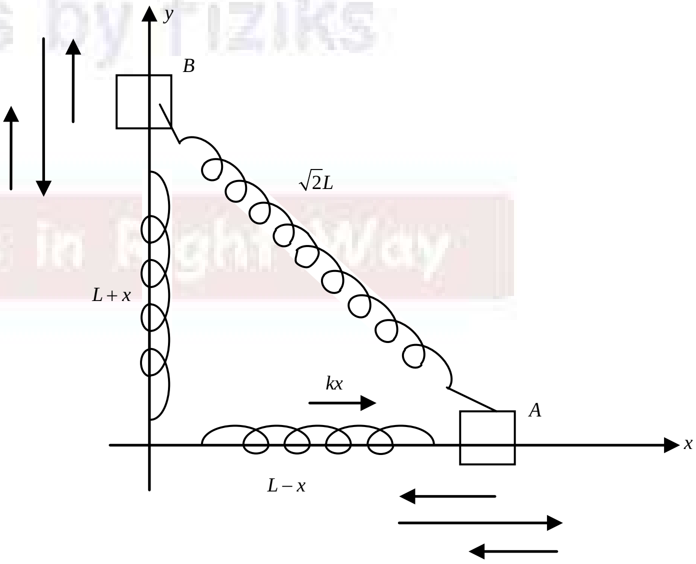
$$\Rightarrow \omega_1=\sqrt{\frac{K+K'}{m}}$$

(2) Total force acting on block A

$$F=-kx \Rightarrow ma=-kx$$

$$\Rightarrow a=-\frac{K}{m}x$$

$$\Rightarrow \omega_2=\sqrt{\frac{k}{m}}$$



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Part-C: 2-Mark Numerical Questions

Q10. A binary star system consists of two stars with same mass M revolving about a common centre of mass in a circular orbit with velocities much smaller than the speed of light, $c = 3.0\times10^{8}\, m/s$. The axis of the plane of rotation is perpendicular to our line of sight. The wavelength of a particular spectral line from one of the stars is observed to change with a period of 2.40×10^{5} seconds. If the ratio of maximum to minimum wavelength of the line is 1.0022, the distance between the stars (in $10^{9}\, m$) to the nearest integer is

Ans: 0025

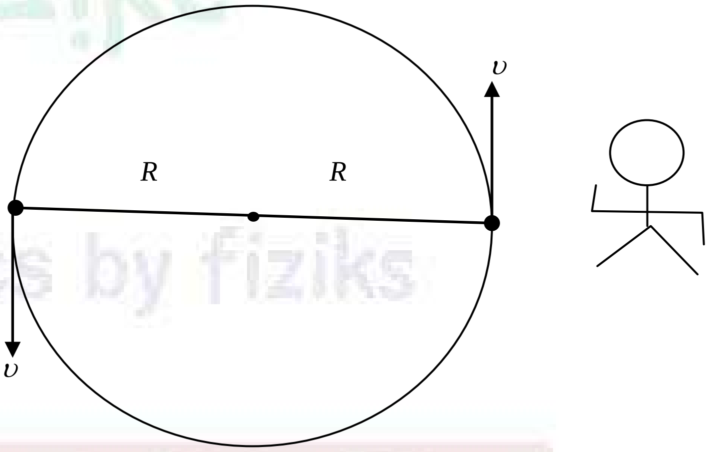
Solution:

For the observer, the observed wavelength will increase (or frequency decrease) when source moves away from him.

$$\lambda'_{\max} = \lambda \frac{c+v}{c} \qquad \text{(Non-relativistic case as } \upsilon \ll c\text{)}$$

The observed wavelength decreases (frequency increases) when source moves toward observer:

$$\lambda'_{\min} = \lambda \frac{c-v}{c}$$



$$\frac{\lambda'_{\max}}{\lambda'_{\min}} = \frac{c+v}{c-v} \Rightarrow 1.0022 = \frac{1+v/c}{1-v/c}$$

$$\Rightarrow \frac{v}{c} = \frac{0.0022}{2.0022}$$

Now $v = R\omega = R\dfrac{2\pi}{T}$

$$R = \frac{vT}{2\pi} = \frac{0.0022\times3\times10^{8}\times2.4\times10^{5}}{2.0022\times2\times3.14}$$

$$= \frac{2.2\times3\times2.4\times10^{10}}{2.0022\times2\times3.14} = 12.59\times10^{9}\,\text{m}$$

Distance between the states $= 2R = 2\times12.59\times10^{9}\,\text{m} = 25.18\times10^{9}\,\text{m}$

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