Physics by fiziks

Learn Physics in Right Way

CSIR NET-JRF Physical Sciences Paper Feb.-2022 Solution

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Part B

Q25. The position of a particle in one dimension changes in discrete steps. With each step it moves to the right, however, the length of the step is drawn from a uniform distribution from the interval $\left[\lambda-\frac{1}{2}w, \lambda+\frac{1}{2}w\right]$, where λ and w are positive constants. If X denotes the distance from the starting point after N steps, the standard deviation $\sqrt{\langle X^2\rangle-\langle X\rangle^2}$ for large values of N is

(a) $\frac{\lambda}{2}\times\sqrt{N}$ (b) $\frac{\lambda}{2}\times\sqrt{\frac{N}{3}}$ (c) $\frac{w}{2}\times\sqrt{N}$ (d) $\frac{w}{2}\times\sqrt{\frac{N}{3}}$

Ans. 25: (d)

Solution:

Let $x_1, x_{2,....} x_N$ denote the length. Hence total length of steps $x = x_1 + x_2 + ... + x_N$

$x_1, x_{2,....} x_N$ are random variables from a uniform distribution from the interval $\left[\lambda-\frac{1}{2}w, \lambda+\frac{1}{2}w\right]$

$$Var[x] = Var(x_1 + x_2 + ... + x_N) = \sum_{i=1}^{N} Var(x_i) + 2\sum_{i<j} Var(x_i, x_j)$$

$\because x_1, x_{2,....} x_N$ are independent random variables; $\therefore \sum_{i<j} Var(x_i, x_j) = 0$

$$\therefore Var[x] = \sum_{i=1}^{N} Var(x_i) = N\frac{\left[\left(\lambda+\frac{1}{2}w\right)-\left(\lambda-\frac{1}{2}w\right)\right]^2}{12} = N\frac{w^2}{12}$$

Hence standard deviation $= \sqrt{Var[x]} = \frac{w}{2}\sqrt{\frac{N}{3}}$

Q26. The volume of the region common to the interiors of two infinitely long cylinders defined by $x^2 + y^2 = 25$ and $x^2 + 4z^2 = 25$ is best approximated by

(a) 225 (b) 333 (c) 423 (d) 625

Ans. 26: (b)

Solution:

$$x^2 + y^2 = 25 \Rightarrow y = \pm\sqrt{25-x^2}; \quad x^2 + 4z^2 = 25 \Rightarrow z = \pm\frac{\sqrt{25-x^2}}{2}$$

In any of the above equations, `x' varies from -5 to 5. Therefore, the volume bounded in the intersecting region is

$$V=\int_{-5}^{5}\int_{-\sqrt{25-x^2}}^{\sqrt{25-x^2}}\int_{-\frac{\sqrt{25-x^2}}{2}}^{\frac{\sqrt{25-x^2}}{2}} dzdydx = 8\int_{0}^{5}\int_{0}^{\sqrt{25-x^2}}\int_{0}^{\frac{\sqrt{25-x^2}}{2}} dzdydx = 8\int_{0}^{5}\int_{0}^{\sqrt{25-x^2}} \frac{\sqrt{25-x^2}}{2}dydx$$

$$V=8\int_0^5 \frac{\sqrt{25-x^2}}{2}\sqrt{25-x^2}\,dx = 4\int_0^5\left(25-x^2\right)dx = 4\left|25x-\frac{x^3}{3}\right|_0^5 = 4\left|125-\frac{125}{3}\right|$$

$V=4\times\frac{250}{3}=333.33$. Hence (b) is correct option.

Q36. A discrete random variable X takes a value from the set $\{-1,0,1,2\}$ with the corresponding probabilities $p(X)=3/10, 2/10, 2/10$ and $3/10$, respectively. The probability distribution $q(Y)=\left(q(0),q(1),q(4)\right)$ of the random variable $Y=X^2$ is

(a) $\left(\frac{1}{5},\frac{3}{5},\frac{1}{5}\right)$ (b) $\left(\frac{1}{5},\frac{1}{2},\frac{3}{10}\right)$ (c) $\left(\frac{2}{5},\frac{2}{5},\frac{1}{5}\right)$ (d) $\left(\frac{3}{10},\frac{3}{10},\frac{2}{5}\right)$

Ans. 36: (b)

Solution: Given that,

$X=\{-1,0,1,2\}$; $p(X)=\{\frac{3}{10},\frac{2}{10},\frac{2}{10},\frac{3}{10}\}$

$X^2=\{0,1,4\}$; $p(X^2)=\{--,--,---\}=?$

For, $X^2=0, X=0 \Rightarrow p(X^2=0)=p(X=0)=\frac{2}{10}$

For, $X^2=1, X=\pm1;\ \Rightarrow p(X^2=1)=p(X=1)+p(X=-1)=\frac{3}{10}+\frac{2}{10}=\frac{1}{2}$

For ($X=-2$ is not in the list),

$X^2=4, X=\pm2 \quad \Rightarrow p(X^2=4)=p(X=2)=\frac{3}{10}$

Thus, $p(X^2)=\{\frac{1}{5},\frac{1}{2},\frac{3}{10}\}$ Hence, (b) is correct option.

Q41. The equation of motion of a one-dimensional forced harmonic oscillator in the presence of a dissipative force is described by $\frac{d^2x}{dt^2}+10\frac{dx}{dt}+16x=6te^{-8t}+4t^2e^{-2t}$. The general form of the particular solution, in terms of constants A,B etc., is

(a) $t\left(At^2+Bt+C\right)e^{-2t}+\left(Dt+E\right)e^{-8t}$ (b) $\left(At^2+Bt+C\right)e^{-2t}+\left(Dt+E\right)e^{-8t}$

(c) $t\left(At^2+Bt+C\right)e^{-2t}+t\left(Dt+E\right)e^{-8t}$ (d) $\left(At^2+Bt+C\right)e^{-2t}+t\left(Dt+E\right)e^{-8t}$

Ans. 41: (c)

Solution:

Given differential equation is $\frac{d^2x}{dt^2}+10\frac{dx}{dt}+16x=6te^{-8t}+4t^2e^{-2t}$

Auxiliary equation is $D^2+10D+16=0 \Rightarrow (D+8)(D+2)=0 \Rightarrow D=-8,-2$

Thus, the complementary function can be written as

$y_{cf}=ae^{-8t}+be^{-2t}$.............(1)

$$PI=\frac{1}{D^2+10D+16}6te^{-8t}+4t^2e^{-2t}=\frac{1}{D^2+10D+16}(6te^{-8t})+\frac{1}{D^2+10D+16}(4t^2e^{-2t}).......(2)$$

$$\frac{1}{D^2+10D+16}(6te^{-8t})=\frac{1}{\cancel{6}}\left[\frac{1}{D+2}-\frac{1}{D+8}\right]\cancel{6}te^{-8t}=\frac{1}{D+2}te^{-8t}-\frac{1}{D+8}te^{-8t}$$

$$=e^{-2t}\int e^{2t}te^{-8t}dt-e^{-8t}\int e^{8t}te^{-8t}dt=e^{-2t}\int te^{-6t}dt-e^{-8t}\int tdt$$

$$=e^{-2t}\left[t\frac{e^{-6t}}{-6}-\int(1)\frac{e^{-6t}}{-6}d\right]-t\frac{t^2}{2}e^{-8t}=e^{-2t}\left[t\frac{e^{-6t}}{-6}-\frac{e^{-6t}}{36}\right]-\frac{t^2}{2}e^{-8t}=t\frac{e^{-8t}}{-6}-\frac{e^{-8t}}{36}-\frac{t^2}{2}e^{-8t}=$$

$$=-\frac{t^2}{2}e^{-8t}-\frac{te^{-8t}}{6}-\frac{e^{-8t}}{36}$$

The last terms in the above expression can be coupled with complementary function

Therefore, $A'=t\left[Dt+E\right]e^{-8t}$...............(3)

$$\frac{1}{D^2+10D+16}(4t^2e^{-2t})=\frac{1}{6}\left[\frac{1}{D+2}-\frac{1}{D+8}\right]4te^{-2t}=\frac{2}{3}\left[\frac{1}{D+2}t^2e^{-2t}-\frac{1}{D+8}t^2e^{-2t}\right]$$

$$=\frac{2}{3}\left[e^{-2t}\int t^2dt-e^{-8t}\int t^2e^{6t}dt\right]=\frac{2}{3}\left[e^{-2t}\frac{t^3}{3}-e^{-8t}\left[\frac{t^2e^{6t}}{6}-\int 2t\frac{e^{6t}}{6}dt\right]\right]$$

$$=\frac{2}{3}\left[e^{-2t}\frac{t^3}{3}-e^{-8t}\left[\frac{t^2e^{6t}}{6}-\frac{1}{3}\int te^{6t}d\right]\right]=\frac{2}{3}\left[e^{-2t}\frac{t^3}{3}-e^{-8t}\left[\frac{t^2e^{6t}}{6}-\frac{1}{3}\left[\frac{te^{6t}}{6}-\int\frac{e^{6t}}{6}dt\right]\right]\right]$$

$$=\frac{2}{3}\left[e^{-2t}\frac{t^3}{3}-e^{-8t}\left[\frac{t^2e^{6t}}{6}-\frac{1}{3}\left[\frac{te^{6t}}{6}-\int\frac{e^{6t}}{6}d\right]\right]\right]=\frac{2}{3}\left[e^{-2t}\frac{t^3}{3}-e^{-8t}\left[\frac{t^2e^{6t}}{6}-\frac{1}{3}\left[\frac{te^{6t}}{6}-\frac{e^{6t}}{3}\right]\cancel{6}\right]\right]$$

$$=\frac{2}{3}\left[e^{-2t}\frac{t^3}{3}-e^{-8t}\left[\frac{t^2e^{6t}}{6}-\frac{te^{6t}}{18}+\frac{e^{6t}}{108}\right]\right]=\frac{2}{3}\left[e^{-2t}\frac{t^3}{3}-\frac{t^2e^{-2t}}{6}+\frac{te^{-2t}}{18}-\frac{e^{-2t}}{108}\right]$$

$$=\frac{2t^3e^{-2t}}{9}-\frac{t^2e^{-2t}}{9}+\frac{te^{-2t}}{27}-\frac{e^{-2t}}{162}$$

The last terms in the above expression can be coupled with complementary function

Therefore, $B'=t\left[At^2+BT+C\right]e^{-2t}$...........(4)

From (3) and (4); $PI=t\left[At^2+BT+C\right]e^{-2t}+t\left[Dt+E\right]e^{-8t}$............(5)

Thus, (c) is correct option.

Q44. A generic 3×3 real matrix A has eigenvalues $0,1$ and 6, and I is the 3×3 identity matrix. The quantity/quantities that cannot be determined from this information is/are the

(a) eigenvalue of $(I+A)^{-1}$ (b) eigenvalue of $(I+A^TA)$

(c) determinant of A^TA (d) rank of A

Ans. 44: (b)

Solution: Given Eigen values are $=0,1,6$

Eigen values of $I+A$ are $=1+$ Eigen values of A

Therefore, Eigen values of $(I+A)^{-1}$ are $=1,\frac{1}{2},\frac{1}{7}$

Therefore, \`a' can be determined

$|A|=0\times1\times6=0$, $|A^T|=0\times1\times6=0$. Therefore, $|AA^T|=|A||A^T|=0$

Thus, (c) can also be determined

As one Eigen value is 0. Therefore, rank is 3-1=2. Hence, \`d' can also be determined.

Thus, \`b' cannot be determined. Hence, it is the correct answer.

Q45. The volume integral $I=\iiint_V \vec{A}\cdot\left(\vec{\nabla}\times\vec{A}\right)d^3x$, is over a region V bounded by a surface Σ (an infinitesimal area element being $\hat{n}ds$, where $\hat{n}$ is the outward unit normal). If it changes to $I+\Delta I$ when the vector $\vec{A}$ is changed to $\vec{A}+\vec{\nabla}\Lambda$, then ΔI can be expressed as

(a) $\iiint_V \vec{\nabla}\cdot\left(\vec{\nabla}\Lambda\times\vec{A}\right)d^3x$ (b) $-\iiint_V \nabla^2\Lambda\, d^3x$

(c) $-\oiint_\Sigma\left(\vec{\nabla}\Lambda\times\vec{A}\right).\hat{n}ds$ (d) $\oiint_\Sigma \vec{\nabla}\Lambda.\hat{n}ds$

Ans. 45: (c)

Solution:

$$I=\int\int\int\vec{A}\cdot(\vec{\nabla}\times\vec{A})d^3x$$

$$I+\Delta I=\int\int\int(\vec{A}+\vec{\nabla}\lambda)\cdot(\vec{\nabla}\times(\vec{A}+\vec{\nabla}\lambda))d^3x$$

$$I+\Delta I=\int\int\int(\vec{A}+\vec{\nabla}\lambda)\cdot(\vec{\nabla}\times(\vec{A})+\vec{\nabla}\times(\vec{\nabla}\lambda))d^3x$$

Now, the curl of the gradient always vanishes.

Therefore, the above equation becomes.

$$I+\Delta I=\int\int\int\left(\vec{A}\cdot\left(\vec{\nabla}\times\vec{A}\right)+\vec{\nabla}\lambda\cdot\left(\vec{\nabla}\times\vec{A}\right)\right)d^3x$$

$$I+\Delta I=\int\int\int\vec{A}\cdot\left(\vec{\nabla}\times\vec{A}\right)d^3x+\int\int\int\vec{\nabla}\lambda\cdot\left(\vec{\nabla}\times\vec{A}\right)d^3x$$

The first term in the above expression is just the I. Thus, we get

$\Delta I = \iiint \vec{\nabla}\lambda \cdot \left(\vec{\nabla}\times\vec{A}\right) d^3x \ldots\ldots\ldots(A)$

We know, $\vec{\nabla}\cdot(\vec{A}\times\vec{B}) = \vec{B}\cdot\left(\vec{\nabla}\times\vec{A}\right) - \vec{A}\cdot\left(\vec{\nabla}\times\vec{B}\right)$

Using, $\vec{A}=\vec{A},\quad \vec{B}=\vec{\nabla}\lambda$ in the above expression, we get

$\vec{\nabla}\cdot(\vec{A}\times\vec{\nabla}\lambda) = \vec{\nabla}\lambda\cdot\left(\vec{\nabla}\times\vec{A}\right) - \vec{A}\cdot\left(\vec{\nabla}\times\vec{\nabla}\lambda\right)$

The second term vanishes again. Therefore, we get $\vec{\nabla}\cdot(\vec{A}\times\vec{\nabla}\lambda) = \vec{\nabla}\lambda\cdot\left(\vec{\nabla}\times\vec{A}\right)$

Substituting, this result in (A), we get $\Delta I = \iiint \vec{\nabla}\cdot(\vec{A}\times\vec{\nabla}\lambda) d^3x$

Using, divergence theorem, we get $\Delta I = \oint\!\!\oint (\vec{A}\times\vec{\nabla}\lambda)\cdot\hat{n}ds = -\oint\!\!\oint (\vec{\nabla}\lambda\times\vec{A})\cdot\hat{n}ds$

Thus, (c) is the correct option.

Part C

Q46. The Newton-Raphson method is to be used to determine the reciprocal of the number $x=4$. If we start with the initial guess 0.20 then after the first iteration the reciprocal is

(a) 0.23 (b) 0.24 (c) 0.25 (d) 0.26

Ans. 46: (b)

Solution: To find the inverse of 4, let $x=\dfrac{1}{4} \Rightarrow f(x)=\dfrac{1}{x}-4=0$

Thus, we need the solution of this equation after first iteration.

Starting point, $x_0 = 0.20$

$$\Rightarrow f(x_0)=\frac{1}{0.20}-4=5-4=1;\ \ f'(x_0)=-\left.\frac{1}{x^2}\right|_{x=x_0}=-\frac{1}{(0.20)^2}=-2$$

$$x_{n+1}=x_n-\frac{f(x_0)}{f'(x_0)}=0.20-\frac{1}{-25}=0.20+0.04=0.24$$

Hence, (b) is correct option.

Q67. The Legendre polynomials $P_n(x), n=0,1,2,\ldots,$ satisfying the orthogonality condition $\int_{-1}^{1} P_n(x)\,p_m(x)\,dx=\dfrac{2}{2n+1}\delta_{nm}$ on the interval $[-1,+1]$, may be defined by the Rodrigues formula $P_n(x)=\dfrac{1}{2^n n!}\dfrac{d^n}{dx^n}\left(x^2-1\right)^n$. The value of the definite integral $\int_{-1}^{1}\left(4+2x-3x^2+4x^3\right)P_3(x)\,dx$ is

(a) $3/5$ (b) $11/15$ (c) $23/32$ (d) $16/35$

Ans. 67: (d)

Solution:

Let

$$4+2x-3x^2+4x^3=a_0P_0(x)+a_1P_1(x)+a_2P_2(x)+a_3P_3(x)$$

$$4+2x-3x^2+4x^3=a_0+a_1x+a_2\left[\frac{3x^2-1}{2}\right]+a_3\left[\frac{5x^3-3x}{2}\right]=\left[a_0-\frac{a_2}{2}\right]+\left[a_1-\frac{3a_3}{2}\right]x+\frac{3}{2}a_2x^2+\frac{5a_3x^3}{2}$$

Comparing, we get

$$a_2\frac{3}{2}=-3\Rightarrow a_2=-2,\frac{5}{2}a_3=4\Rightarrow a_3=\frac{8}{5}$$

$$a_0-\frac{a_2}{2}=4\Rightarrow a_0=4+\frac{-2}{2}=3$$

$$a_1-\frac{3a_3}{2}=2\Rightarrow a_1=\frac{24}{10}+2=\frac{12}{5}+2=\frac{22}{5}$$

Therefore,

$$\int_{-1}^{1}(4+2x-3x^2+4x^3)P_3(x)dx=\int_{-1}^{1}\left(3P_0(x)+\frac{22}{5}P_1(x)-2P_2(x)+\frac{8}{5}P_3(x)\right)P_3(x)dx=$$

$$\int_{-1}^{1}\frac{8}{5}P_3(x)P_3(x)dx=\frac{8}{5}\int_{-1}^{1}P_3(x)P_3(x)dx=\frac{8}{5}\frac{2}{2\times 3+1}=\frac{16}{3}$$

Other integrals vanish because of orthogonal property. Thus, (d) is correct option.

Q68. If we use the Fourier transform $\phi(x,y)=\int e^{ikx}\phi_k(y)dk$ to solve the partial differential equation $-\frac{\partial^2\phi(x,y)}{\partial y^2}-\frac{1}{y^2}\frac{\partial^2\phi(x,y)}{\partial x^2}+\frac{m^2}{y^2}\phi(x,y)=0$ in the half-plane $\{(x,y):-\infty<x<\infty,0<y<\infty\}$ the Fourier modes $\phi_k(y)$ depend on y as y^{α} and y^{β}. The value of α and β are

(a) $\frac{1}{2}+\sqrt{1+4(k^2+m^2)}$ and $\frac{1}{2}-\sqrt{1+4(k^2+m^2)}$

(b) $1+\sqrt{1+4(k^2+m^2)}$ and $1-\sqrt{1+4(k^2+m^2)}$

(c) $\frac{1}{2}+\frac{1}{2}\sqrt{1+4(k^2+m^2)}$ and $\frac{1}{2}-\frac{1}{2}\sqrt{1+4(k^2+m^2)}$

(d) $1+\frac{1}{2}\sqrt{1+4(k^2+m^2)}$ and $1-\frac{1}{2}\sqrt{1+4(k^2+m^2)}$

Ans. 68: (c)

Solution:

$$\phi(x,y)=\int e^{ikx}\phi_k(y)dk \Rightarrow \phi_k(y)=\int e^{-ikx}\phi(x,y)dx$$

Given equation

$$-\frac{\partial^2\phi(x,y)}{\partial y^2}-\frac{1}{y^2}\frac{\partial^2\phi(x,y)}{\partial x^2}+\frac{m^2}{y^2}\phi(x,y)=0 \Rightarrow -y^2\frac{\partial^2\phi(x,y)}{\partial y^2}-\frac{\partial^2\phi(x,y)}{\partial x^2}+m^2\phi(x,y)=0$$

Multiplying both sides by e^{-ikx} and integrating with respect to `x', we get

$$-y^2\frac{\partial^2}{\partial y^2}\left[\int e^{-ikx}\phi(x,y)dx\right]-\int e^{-ikx}\frac{\partial^2\phi(x,y)}{\partial x^2}dx+m^2\int e^{-ikx}\phi(x,y)dx=0.............(1)$$

Using $\int e^{-ikx}\frac{\partial^2\phi(x,y)}{\partial x^2}dx=(-ik)^2\int e^{-ikx}\phi(x,y)dx$ in (1), we get

$$-y^2\frac{\partial^2\phi_k(y)}{\partial y^2}-(-ik)^2\phi_k(y)+m^2\phi_k(y)=0$$

$$\Rightarrow y^2\frac{\partial^2\phi_k(y)}{\partial y^2}-(k^2+m^2)\phi_k(y)=0..............(2)$$

Using, $y=e^z$, (2) becomes $\left[D(D-1)-(k^2+m^2)\right]\phi_k(z)=0...........(3)$

The auxiliary equation can be written as $D^2-D-(k^2+m^2)=0 \Rightarrow D=\frac{1\pm\sqrt{1+4(k^2+m^2)}}{2}$

The solution of (3) can therefore be written as

$$\phi_k(z)=ae^{\frac{1+\sqrt{1+4(k^2+m^2)}}{2}z}+be^{\frac{1-\sqrt{1+4(k^2+m^2)}}{2}z} \Rightarrow \phi_k(z)=a(e^z)^{\frac{1+\sqrt{1+4(k^2+m^2)}}{2}}+b(e^z)^{\frac{1-\sqrt{1+4(k^2+m^2)}}{2}}$$

Reverting to original variable, we get $\phi_k(y)=a(y)^{\frac{1+\sqrt{1+4(k^2+m^2)}}{2}}+b(y)^{\frac{1-\sqrt{1+4(k^2+m^2)}}{2}}$

Therefore, $\alpha=\frac{1+\sqrt{1+4(k^2+m^2)}}{2}$ and $\beta=\frac{1-\sqrt{1+4(k^2+m^2)}}{2}$ Thus, (c) is correct option.

Part B

Q22. A particle in one dimension executes oscillatory motion in a potential $V(x) = A|x|$, where $A > 0$ is a constant of appropriate dimension. If the time period T of its oscillation depends on the total energy E as E^a, then the value of a is

(a) $1/3$ (b) $1/2$ (c) $2/3$ (d) $3/4$

Ans. 22: (b)

Solution: Total energy $E = \dfrac{p^2}{2m} + A|x|$

Action angle variable $J = 4\int_0^{E/A} \sqrt{2m(E - A|x|)}dx = 4\sqrt{2mE}\int_0^{E/A}\sqrt{1 - \dfrac{A}{E}x}\, dx$

For $0 \le x \le \dfrac{E}{A} \quad \to |x| = x$

Let $\dfrac{A}{E}x = t \to dx = \dfrac{E}{A}dt \quad ; J = 4\sqrt{2mE}\,\dfrac{E}{A}\int_0^1 \sqrt{1-t}dt \Rightarrow J = x_0 E^{3/2}$

Time period $T = \dfrac{\partial J}{\partial E} = \dfrac{3}{2}x_0 E^{1/2} \Rightarrow \alpha = \dfrac{1}{2}$

Q24. A particle of mass $1GeV/c^2$ and its antiparticle, both moving with the same speed v, produce new particle x of mass $10GeV/c^2$ in a head on collision. The minimum value of v required for this process is closest to

(a) $0.83c$ (b) $0.93c$

(c) $0.98c$ (d) $0.88c$

Ans. 24: (c)

Solution:

υ $\quad$ υ

Particle $m = 1\dfrac{GeV}{c^2}$ $\quad$ Anti-Particle $m = 1\dfrac{GeV}{e^2}$ $\quad$ x $m = 10\dfrac{GeV}{c^2}$

Conservation of energy $\dfrac{mc^2}{\sqrt{1-\dfrac{v^2}{c^2}}} + \dfrac{mc^2}{\sqrt{1-\dfrac{v^2}{c^2}}} = Mc^2 \Rightarrow \dfrac{2m\cancel{c^2}}{\sqrt{1-\dfrac{v^2}{c^2}}} = M\cancel{c^2}$

$$2\times 1 = 10\sqrt{1-\frac{v^2}{c^2}} \Rightarrow \frac{1}{25} = 1 - \frac{v^2}{c^2} \Rightarrow v = \frac{\sqrt{24}}{5}c = 0.93c$$

Q29. A monochromatic source emitting radiation with a certain frequency moves with a velocity v away from a stationary observer A. It is moving towards another observer B (also at rest) along a line joining the two. The frequencies of the radiation recorded by A and B are V_A and V_B, respectively. If the ratio $\frac{V_B}{V_A}=7$, then the value of v/c is

(a) $1/2$ (b) $1/4$ (c) $3/4$ (d) $\sqrt{3}/2$

Ans. 29: (c)

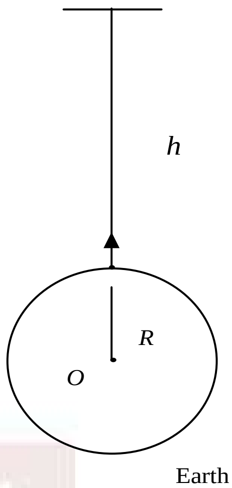
Solution.

$$\upsilon_A=\upsilon_A\sqrt{\frac{c-\upsilon}{c+\upsilon}};\ \upsilon_B=\upsilon_0\sqrt{\frac{c+\upsilon}{c-\upsilon}};\quad \frac{\upsilon_B}{\upsilon_A}=\frac{c+\upsilon}{c-\upsilon}=\frac{1+\upsilon/c}{1-\upsilon/c}=7$$

$$7-7\frac{\upsilon}{c}=1+\frac{\upsilon}{c}\Rightarrow 8\frac{\upsilon}{c}=6\Rightarrow \boxed{\frac{\upsilon}{c}=\frac{3}{4}}$$

Q30. A particle, thrown with a speed v from the earth's surface, attains a maximum height h (measured from the surface of the earth). If v is half the escape velocity and R denotes the radius of earth, then h/R is

(a) $2/3$ (b) $1/3$ (c) $1/4$ (d) $1/2$



Ans. 30: (b)

Solution.: $\upsilon=\frac{1}{2}\upsilon_e=\frac{1}{2}\sqrt{\frac{2GM}{R}}$. Here M is the mass of the earth.

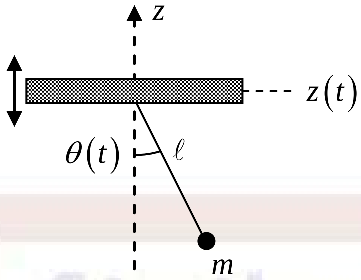
Conservation of mechanical energy

$$-\frac{GMm}{R}+\frac{1}{2}m\upsilon^2=-\frac{GMm}{(R+h)}+0\Rightarrow -\frac{GMm}{R}+\frac{GMm}{4R}=-\frac{GMm}{R+h}$$

$$\Rightarrow -\frac{3GMm}{4R}=-\frac{GMm}{R+h}\Rightarrow 3R+3h=4R\Rightarrow 3h=R\ \Rightarrow \boxed{\frac{h}{R}=\frac{1}{3}}$$

Part C

Q48. The fulcrum of a simple pendulum (consisting of a particle of mass m attached to the support by a massless string of length ℓ) oscillates vertically as $\sin z(t) = a\sin\omega t$, where ω is a constant. The pendulum moves in a vertical plane and $\theta(t)$ denotes its angular position with respect to the z-axis



If $\ell\dfrac{d^2\theta}{dt^2}+\sin\theta\left(g-f(t)\right)=0$ (where g is the acceleration due to gravity) describes the equation of motion of the mass, then $f(t)$ is

(a) $a\omega^2\cos\omega t$ (b) $a\omega^2\sin\omega t$ (c) $-a\omega^2\cos\omega t$ (d) $-a\omega^2\sin\omega t$

Ans. 48: (b)

Solution:

$x = l\sin\theta$, $z = l\cos\theta + z(t) = l\cos\theta + a\sin\omega t \;\Rightarrow x = l\cos\theta\dot{\theta}$, $\dot{z} = -l\sin\theta\dot{\theta} + a\omega\cos\omega t$

$$L = T - V = \frac{1}{2}m\left[\dot{x}^2+\dot{z}^2\right]-(-mgz)$$

$$L = \frac{1}{2}m\left[l^2\dot{\theta}^2 + a^2\omega^2\cos^2\omega t + 2al\omega\sin\theta\cos\omega t\dot{\theta}\right] + mg\left(l\cos\theta + a\sin\omega t\right)$$

$$\Rightarrow \frac{\partial L}{\partial\dot{\theta}} = ml^2\dot{\theta} + mal\omega\sin\theta\cos\omega t \text{ and } \frac{\partial L}{\partial\theta} = mal\omega\cos\theta\cos\omega t\dot{\theta} - mgl\sin\theta$$

$$\Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}}\right) = ml^2\ddot{\theta} + mal\omega\cos\theta\cos\omega t\dot{\theta} - mal\omega^2\sin\theta\sin\omega t$$

$$\because \frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}}\right) - \frac{\partial L}{\partial\theta} = 0 \Rightarrow ml^2\ddot{\theta} - mal\omega^2\sin\theta\sin\omega t + mgl\sin\theta = 0$$

$$\Rightarrow l\frac{d^2\theta}{dt^2} + \sin\theta\left[g - a\omega^2\sin\omega t\right] = 0 \Rightarrow l\frac{d^2\theta}{dt^2} + \sin\theta\left[g - f(t)\right] = 0$$

where $f(t) = a\omega^2\sin\omega t$

Q57. A particle in two dimensions is found to trace an orbit $r(\theta)=r_0\theta^2$. If it is moving under the influence of a central potential $V(r)=c_1r^{-a}+c_2r^{-b}$, where r_0, c_1 and c_2 are constants of appropriate dimensions, the values of a and b, respectively, are

(a) 2 and 4 (b) 2 and 3 (c) 3 and 4 (d) 1 and 3

Ans. 57: (b)

Solution: $u=\frac{1}{r}=\frac{1}{r_0\theta^2}\Rightarrow\frac{du}{d\theta}=-\frac{2}{r_0\theta^3}\Rightarrow\frac{d^2u}{d\theta^2}=\frac{6}{r_0\theta^4}$

Differential equation of the orbit $\frac{d^2u}{d\theta^2}+u=-\frac{m}{\ell^2u^2}f\left(\frac{1}{u}\right)\Rightarrow\frac{6}{r_0\theta^4}+\frac{1}{r_0\theta^2}=-\frac{mr_0^2\theta^4}{\ell^2}f\left(\frac{1}{u}\right)$

$f\left(\frac{1}{u}\right)=-\frac{6\ell^2}{mr_0^3\theta^8}-\frac{\ell^2}{mr_0^3\theta^6}\quad\Rightarrow f(r)=-Ar^{-4}-Br^{-3}$ where A and B are constants

$A=\frac{6\ell^2r_0}{m}$ and $B=\frac{\ell^2}{m}$

$V(r)=-\int f(r)dr=\int\left[Ar^{-4}+Br^{-3}\right]dr=A\frac{r^{-4+1}}{-3}+B\frac{r^{-3+1}}{-2}$

$V(r)=c_1r^{-3}+c_2r^{-2}=c_1r^{-a}+c_2r^{-b}\qquad\Rightarrow a=3,\ b=2$

Q58. A particle of mass m moves in a potential that is $V=\frac{1}{2}m\left(\omega_1^2x^2+\omega_2^2y^2+\omega_3^2z^2\right)$ in the coordinates of a non-inertial frame F. The frame F is rotating with respect to an inertial frame with an angular velocity $\hat{k}\Omega$, where $\hat{k}$ it is the unit vector along their common z-axis. The motion of the particle is unstable for all angular frequencies satisfying

(a) $\left(\Omega^2-\omega_1^2\right)\left(\Omega^2-\omega_2^2\right)>0$ (b) $\left(\Omega^2-\omega_1^2\right)\left(\Omega^2-\omega_2^2\right)<0$

(c) $\left(\Omega^2-(\omega_1+\omega_2)^2\right)\left(\Omega^2-|\omega_1-\omega_2|^2\right)>0$ (d) $\left(\Omega^2-(\omega_1+\omega_2)^2\right)\left(\Omega^2-|\omega_1-\omega_2|^2\right)<0$

Ans. 58: (b)

Q60. A satellite of mass m orbits around earth in an elliptic trajectory of semi-major axis a. At a radial distance $r=r_0$, measured from the centre of the earth, the kinetic energy is equal to half the magnitude of the total energy. If M denotes the mass of the earth and the total energy is $-\frac{GMm}{2a}$, the value of r_0/a is nearest to

(a) 1.33 (b) 1.48 (c) 1.25 (d) 1.67

Ans. 60: (a)

Solution: $TE = -\frac{GMm}{2a}, \quad KE = \frac{1}{2}|TE| = \frac{GMm}{4a}$

$PE = TE - KE = -\frac{GMm}{2a} - \frac{GMm}{4a} \quad \Rightarrow PE = -\frac{3GMm}{4a}$ ....(1)

The potential energy at $r = r_0$ will be $PE = -\frac{GMm}{r_0}$ ....(2)

From Eqs. (1) and (2); $-\frac{3GMm}{4a} = -\frac{GMm}{r_0} \Rightarrow \frac{r_0}{a} = \frac{4}{3} = 1.33$

Part B

Q23. The components of the electric field, in a region of space devoid of any change or current sources, are given to be $E_i = a_i + \sum_{j=1,2,3} b_{ij} x_j$, where a_i and b_{ij} are constants independent of the coordinates. The number of independent components of the matrix b_{ij} is

(a) 5 (b) 6 (c) 3 (d) 4

Ans. 23: (a)

Solution.: $E_i = a_i + \sum_{j=1}^{3} b_{ij} x_j$ .This equation represents a set of three equations

$$\left.\begin{aligned} E_1 &= a_1 + b_{11}x_1 + b_{12}x_2 + b_{13}x_3 \\ E_2 &= a_2 + b_{21}x_1 + b_{22}x_2 + b_{23}x_3 \\ E_3 &= a_3 + b_{31}x_1 + b_{32}x_2 + b_{33}x_3 \end{aligned}\right] - (1)$$

Let $E_1 = E_x, E_2 = E_y, E_3 = E_z$ and $x_1 = x, x_2 = y, x_3 = z, (1)$ can be written as

$$\left.\begin{aligned} E_x &= a_1 + b_{11}x + b_{12}y + b_{13}z \\ E_y &= a_2 + b_{21}x + b_{22}y + b_{23}z \\ E_z &= a_3 + b_{31}x + b_{32}y + b_{33}z \end{aligned}\right] - (2)$$

For electrostatic field, $\vec{\nabla} \times \vec{E} = 0 \Rightarrow \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_x & E_y & E_z \end{vmatrix} = 0$

Let's just look at the x-component (which will be equal to zero). $\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} = 0 - (3)$

Using (2); $\frac{\partial E_z}{\partial y} = b_{32}, \frac{\partial E_y}{\partial z} = b_{23}$. From (3); $b_{32} - b_{23} = 0 \Rightarrow b_{32} = b_{23}$

Similarly, we will get from 'y' and 'z' components $b_{13} = b_{31}, b_{21} = b_{12}$

Thus, it means b_{ij} is symmetric. A symmetric matrix will have,

3 diagonal +3 off- diagonal = 6-independent components $-(4)$

Also, as the region is charge free, therefore $\vec{\nabla} \cdot \vec{E} = 0 \Rightarrow \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = 0 - (5)$

From (2), $\frac{\partial E_x}{\partial x} = b_{11}, \frac{\partial E_y}{\partial y} = b_{22}, \frac{\partial E_z}{\partial z} = b_{33}$. Putting, this result in (5), we get

$b_{11} + b_{22} + b_{33} = 0$. This implies that atleast one of the diagonal elements is dependent. Therefore, the total number of independent components = 6-1=5. Therefore, (a) is correct option.

Q37. In an experiment to measure the charge to mass ratio e/m of the electron by Thomson's method, the values of the deflecting electric field and the accelerating potential are $6\times10^{6}\,N/C$ (newton per coulomb) and $150V$, respectively. The magnitude of the magnetic field that leads to zero deflection of the electron beam is closest to

(a) $0.6T$ (b) $1.2T$ (c) $0.4T$ (d) $0.8T$

Ans. 37: (d)

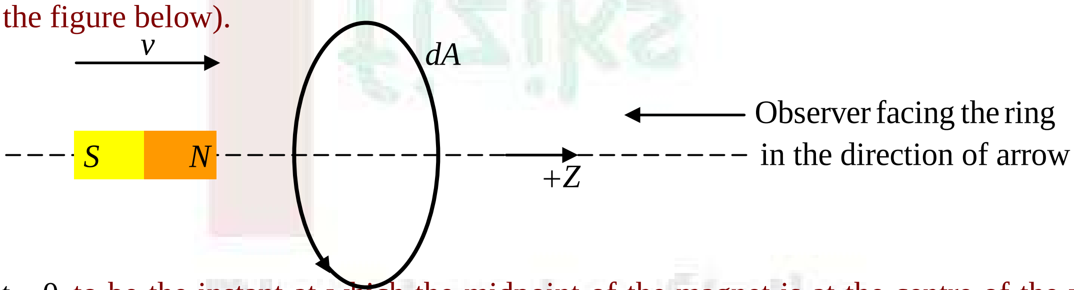
Solution.: Let's determine the velocity of an electron accelerated to 150V.

Using, the classical formula relating kinetic energy and accelerating potential,

$$\frac{1}{2}mv^{2}=eV\Rightarrow v=\sqrt{\frac{2\times150\times1.6\times10^{-19}}{9.1\times10^{-31}}}=7.26\times10^{6}\,m/s$$

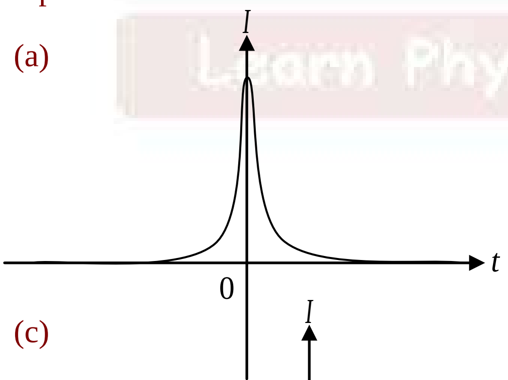
For, zero deflection, $B\cancel{e}v=\cancel{e}V\Rightarrow B=\frac{E}{v}=\frac{6\times10^{6}}{7.26\times10^{6}}\cong0.83T$. Thus, (d) is the correct option.

Q39. A conducting wire in the shape of a circle lies on the (x,y)-plane with its centre at the origin. A bar magnet moves with a constant velocity towards the wire along the z-axis (as shown in the figure below).

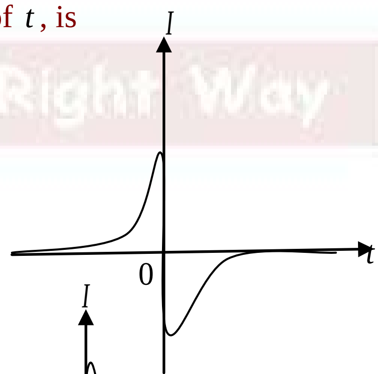


We take $t=0$ to be the instant at which the midpoint of the magnet is at the centre of the wire loop and the induced current to be positive when it is counter-clockwise as viewed by the observer facing the loop and the incoming magnet. In these conventions, the best schematic representation of the induced current $I(t)$ as a function of t, is

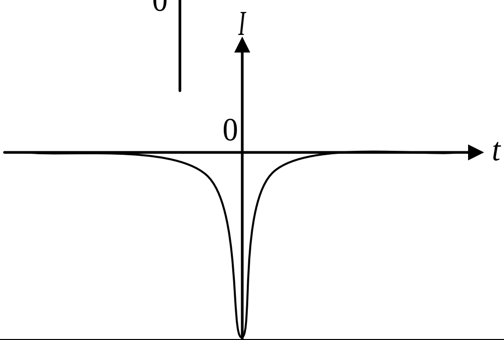
(a)



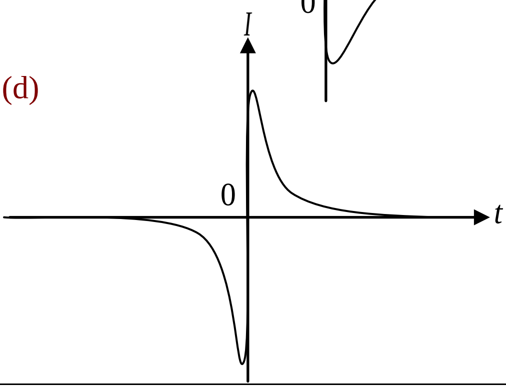
(b)



(c)

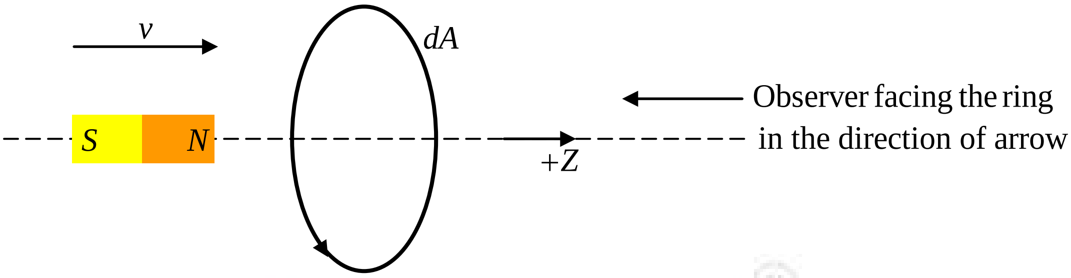


(d)



Ans. 39: (d)

Solution: As, north pole of the magnet moves towards the coil, the induced current must flow in a direction (clockwise as seen from right) to create north polarity on left. However, the current seen from right and flowing counterclockwise is to be considered positive.



This current will produce a south polarity on the left of the coil. Thus, as seen by an observer from right, the current must flow clockwise to produce a north polarity on left. This clockwise current will be negative. Thus, as the bar magnet approaches the coil, first induced current will be negative and after it is about to cross, induced current must be positive. Thus, option (d) should be the correct answer.

Q40. The vector potential for an almost point like magnetic dipole located at the origin is $\vec{A} = \frac{\mu \sin\theta}{4\pi r^2}\hat{\phi}$ where (r, θ, ϕ) denote the spherical polar coordinates and $\hat{\phi}$ is the unit vector along $\hat{\phi}$. A particle of mass m and charge q, moving in the equatorial plane of the dipole, starts at time $= t = 0$ with an initial speed v_0 and an impact parameter b. Its instantaneous speed at the point of closest approach is

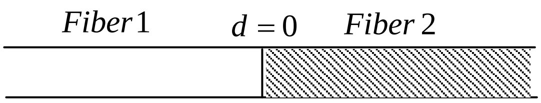
(a) v_0 (b) $0/0$ (c) $v_0 + \frac{\mu q}{4\pi m b^2}$ (d) $\sqrt{v_0^2 + \left(\frac{\mu q}{4\pi m b^2}\right)^2}$

Ans. 40: (a)

Solution: A static magnetic field does not alter the magnitude of speed of a charged particle. It only alters the direction of motion. Hence, its speed will be the same as the one it started with. (i.e., v_0). Thus, (a) is the correct answer.

Part C

Q47. A laser beam propagates from fiber 1 to fiber 2 in a cavity made up of two optical fibers (as shown in the figure). The loss factor of fiber 2 is $10\,dB/km$.



If $E_2(d)$ denotes the magnitude of the electric field in fiber 2 at a distance d from the interface, the ratio $E_2(0)/E_2(d)$ for $d=10\,km$, is

(a) 10^2 (b) 10^3 (c) 10^5 (d) 10^7

Ans. 47: (c)

Solution.: As attenuation leads to a loss of power along the fiber, the output power is significantly less than the couples power. Let the couples optical power is $P(0)$ i.e at origin $(z=0)$. Then the power at a distance z is given by $P(z)=P(0)e^{-\alpha_p z}$ where α_p is fiber attenuation constant (per km). $\alpha_p=\frac{1}{z}\ln\left[\frac{P(0)}{P(z)}\right]\Rightarrow \alpha_{dB/km}=10\frac{1}{z}\ln\left[\frac{P(0)}{P(z)}\right]$

$$10=10\frac{1}{10}\log\frac{P(0)}{P(d)}\Rightarrow\frac{P(0)}{P(d)}=10^{10}\Rightarrow\frac{E(0)}{E(d)}=\sqrt{10^{10}}=10^5$$

Q54. A perfectly conducting fluid of permittivity ε and permeability μ flows with a uniform velocity $\vec{v}$ in the presence of time dependent electric and magnetic fields $\vec{E}$ and $\vec{B}$, respectively, if there is a finite current density in the fluid, then

(a) $\vec{\nabla}\times\left(\vec{v}\times\vec{B}\right)=\frac{\partial\vec{B}}{\partial t}$ (b) $\vec{\nabla}\times\left(\vec{v}\times\vec{B}\right)=-\frac{\partial\vec{B}}{\partial t}$

(c) $\vec{\nabla}\times\left(\vec{v}\times\vec{B}\right)=\sqrt{\varepsilon\mu}\frac{\partial\vec{E}}{\partial t}$ (d) $\vec{\nabla}\times\left(\vec{v}\times\vec{B}\right)=-\sqrt{\varepsilon\mu}\frac{\partial\vec{E}}{\partial t}$

Ans. 54: (a)

Solution: The generalised Ohm's law for conducting fluids is given by $\vec{J}=\sigma\vec{E}+\sigma\vec{v}\times\vec{B}$

If, there is no net current, $\vec{J}=0$. Thus, the above equation becomes,

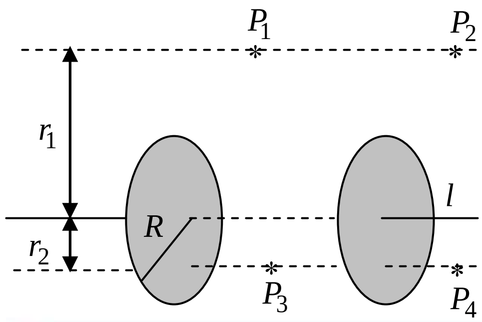
$\vec{E}+\vec{v}\times\vec{B}=0$ $\qquad\sigma$ being common, cancel's out.

Taking curl of the above equation, we get $\vec{\nabla}\times\left(\vec{E}+\vec{v}\times\vec{B}\right)=\vec{\nabla}\times\vec{E}+\vec{\nabla}\times\left(\vec{v}\times\vec{B}\right)=0$

Using $\vec{E}\times\vec{E}=-\frac{\partial\vec{B}}{\partial t}$ in the above equation, we get $-\frac{\partial\vec{B}}{\partial t}+\vec{\nabla}\times\left(\vec{v}\times\vec{B}\right)=0\Rightarrow\vec{\nabla}\times\left(\vec{v}\times\vec{B}\right)=\frac{\partial\vec{B}}{\partial t}$

Thus, (a) is the correct answer.

Q63. The figure below shows an ideal capacitor consisting of two parallel circular plates of radius R. Points P_1 and P_2 are at a transverse distance, $r_1 > R$ from the line joining the centers of the plates, while points P_3 and P_4 are at a transverse distance $r_2 < R$.



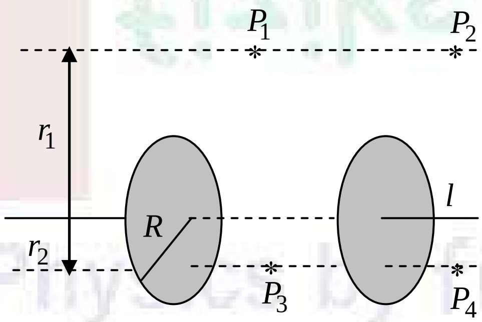
It $B(x)$ denotes the magnitude of the magnetic fields at these points, which of the following holds while the capacitor is charging?

(a) $B(P_1) < B(P_2)$ and $B(P_3) < B(P_4)$ (b) $B(P_1) > B(P_2)$ and $B(P_3) > B(P_4)$

(c) $B(P_1) = B(P_2)$ and $B(P_3) < B(P_4)$ (d) $B(P_1) = B(P_2)$ and $B(P_3) > B(P_4)$

Ans. 63: (c)

Solution:



Magnetic field at P_2 and P_4 can be simply written using Ampere's law (as these points are outside the capacitor, therefore magnetic field only depends upon the magnitude of free current which is just I).

Thus, $B_2(r) = \dfrac{\mu_0 I}{2\pi r_1}$ and $B_4(r) = \dfrac{\mu_0 I}{2\pi r_2}$

At P_1 and P_3, magnetic field depends upon displacement current.

Field at P_1: -

Using $\oint \vec{B}\cdot\vec{dl} = \mu_0 I_{enc} + \mu_0\varepsilon_0 \int_S \dfrac{d\vec{E}}{dt}.d\vec{a}$, $E = \dfrac{\sigma}{\varepsilon_0}, A = \pi R^2$

The conduction current is zero. Further, note that the displacement current does not flow outside the plates, therefore $r = R$ on R.H.S and $r = r_1$ on L.H.S.

Thus, we get $B_1 \times 2\pi r_1 = \mu_0\varepsilon_0 \dfrac{d}{dt}\left(\dfrac{\sigma}{\varepsilon_0}\pi R^2\right) = \dfrac{\mu_0\varepsilon_0}{\varepsilon_0}\dfrac{d}{dt}(q) = \mu_0 I$, $\left(\pi R^2\sigma = q\right) \Rightarrow B_1 = \dfrac{\mu_0 I}{2\pi r_1}$

Field at P_3: -

Using $\oint \vec{B}\cdot\vec{dl} = \mu_0 I_{enc} + \mu_0\varepsilon_0 \int_S \dfrac{d\vec{E}}{dt}.d\vec{a}$, $E = \dfrac{\sigma}{\varepsilon_0}, A = \pi r_2^2$

Note that displacement current flowing through only $r = r_2$ counts on R.H.S. Therefore $r = r_2$ on R.H.S as well as on L.H.S.

Thus, we get

$$B_3 \times 2\pi r_2 = \mu_0\varepsilon_0 \frac{d}{dt}\left(\frac{\sigma}{\varepsilon_0}\pi r_2^2\right) = \mu_0\varepsilon_0 \frac{d}{dt}\left(\frac{q}{\pi R^2\varepsilon_0}\pi r_2^2\right) = \mu_0\varepsilon_0 \frac{d}{dt}(q)\frac{\pi r_2^2}{\pi R^2\varepsilon_0}, \quad \left(\sigma = \frac{q}{\pi R^2}, I = \frac{dq}{dt}\right)$$

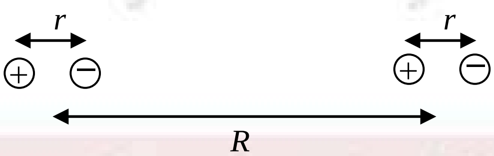
$$\Rightarrow B_3 \times 2\pi r_2 = \mu_0 I \frac{r_2^2}{R^2} \Rightarrow B_3 = \frac{\mu_0 I r_2}{2\pi R^2}$$

Comparing, $B_1 = B_2$

and $\dfrac{B_3}{B_4} = \dfrac{\mu_0 I r_2}{2\pi R^2}\dfrac{2\pi r_2}{\mu_0 I} = \dfrac{r_2^2}{R^2} < 1 \left(\because r_2 < R\right) \qquad \Rightarrow B_3 < B_4$

Thus, (c) is the correct answer.

Q66. A linear diatomic molecule consists of two identical small electric dipoles with an equilibrium separation R, which is assumed to be a constant. Each dipole has charges $\pm q$ of mass m separated by r when the molecule is at equilibrium. Each dipole can execute simple harmonic motion of angular frequency ω



Recall that the interaction potential between two dipoles of moments $\vec{p}_1$ and $\vec{p}_2$, separated by $\vec{R}_{12} = R_{12}\hat{n}$ is $\left(\vec{p}_1\cdot\vec{p}_2 - 3\left(\vec{p}_1\cdot\hat{n}\right)\left(\vec{p}_2\cdot\hat{n}\right)/\left(4\pi\epsilon_0 R_{12}^3\right)\right.$.

Assume that $R \gg r$ and let $\Omega^2 = \dfrac{q^2}{4\pi\epsilon_0 mR^3}$. The angular frequencies of small oscillations of the diatomic molecule are

(a) $\sqrt{\omega^2 + \Omega^2}$ and $\sqrt{\omega^2 - \Omega^2}$

(b) $\sqrt{\omega^2 + 3\Omega^2}$ and $\sqrt{\omega^2 - 3\Omega^2}$

(c) $\sqrt{\omega^2 + 4\Omega^2}$ and $\sqrt{\omega^2 - 4\Omega^2}$

(d) $\sqrt{\omega^2 + 2\Omega^2}$ and $\sqrt{\omega^2 - 2\Omega^2}$

Ans. 66: (c)

Solution:

We need to remember that for two coupled oscillators (two equal masses attached by a spring of force constant κ and attached to the walls from two sides with a spring of force constant k), the difference of squares of allowed frequency of oscillations is given by

$$\omega_2^2 - \omega_1^2 = 2\frac{\kappa}{m}, \quad \omega_1 = \sqrt{\frac{k}{m}}..............(1)$$

The situation here is identical. The interaction energy of two dipoles which are parallel is given by (given in the statement of the problem and taking the angle between the parallel dipoles to be zero degree)

$$= \frac{-2p^2}{4\pi\varepsilon_0 R^3}, p_1 = p_2 = p = qr \qquad \Rightarrow U = \frac{-2q^2r^2}{4\pi\varepsilon_0 R^3}$$

$$F = -\frac{\partial U}{\partial r} = \frac{-4q^2 r}{4\pi\varepsilon_0 R^3} = -\kappa r \qquad \left(\kappa = \frac{4q^2}{4\pi\varepsilon_0 R^3}\right)$$

Therefore, substituting the value of force constant obtained above in the (1), we get

$$\omega_2^2 - \omega_1^2 = \frac{2}{m}\frac{4q^2}{4\pi\varepsilon_0 R^3} = \frac{8q^2}{4\pi\varepsilon_0 m R^3} = 8\Omega^2, \qquad \left(\Omega^2 = \frac{q^2}{4\pi\varepsilon_0 m R^3}\right)$$

The value of Ω^2 is given in the statement of the problem. This is the difference expected in the two frequencies. If we look for this difference of frequencies in the given options, only (c) satisfies this criterion. Therefore, it is the correct option.

Part B

Q21. Which of the following two physical quantities cannot be measured simultaneously with arbitrary accuracy for the motion of a quantum particle in three dimensions?

(a) square of the radial position and z-component of angular momentum (r^2 and L_z)

(b) x- components of linear and angular momenta (p_x and L_x)

(c) y-component of position and z-component of angular momentum (y and L_x)

(d) squares of the magnitudes of the linear and angular momenta (p^2 and L^2)

Ans. 21: (c)

Solution: The two physical quantities cannot be measured simultaneously with arbitrary accuracy in quantum mechanics whose commutator is not zero.

(a) $\left[r^2, L_z\right]=\left[x^2+y^2+z^2, L_z\right]=\left[x^2, L_z\right]+\left[y^2, L_z\right]+\left[z^2, L_z\right]$

$$=x\left[x, L_z\right]+\left[x, L_z\right]x+y\left[y, L_z\right]+\left[y, L_z\right]y=x(-i\hbar y)+(-i\hbar yx)+y(i\hbar x)+(i\hbar x)y=0$$

where, we have used $\left[x, L_y\right]=-i\hbar y:\left[y, L_z\right]=i\hbar x\,;\left[z, L_z\right]=0$

(b) $\left[p_x, L_x\right]=\left[p_x, yp_z, zp_y\right]=\left[p_x, yp_z\right]-\left[p_x-zp_y\right]$

$$=y\left[p_x, p_z\right]+\left[p_x, y\right]p_z+z\left[p_x, p_z\right]-\left[p_x, z\right]p_y$$

$\left[p_x, L_x\right]=0$

where, we have used $\left[p_x, p_z\right]=\left[p_x, y\right]=\left[p_x, z\right]=0$

(c) $\left[p^2, L^2\right]=\left[p^2, r^2p^2-(\vec{r}\cdot\vec{p})^2+i\hbar(\vec{r}\cdot\vec{p})\right]$

$$=\left[p^2, r^2p^2\right]-\left[p^2,(\vec{r}\cdot\vec{p})^2\right]+i\hbar\left[p^2,(\vec{r}\cdot\vec{p})\right]=0$$

where, we have used $\left[p, r\right]=0$

(d) $\left[y, L_z\right]=\left[y, xp_y-yp_x\right]=\left[y, xp_y\right]-\left[y, yp_x\right]=x\left[y, p_y\right]+\left[y, x\right]p_y-y\left[y, p_x\right]-\left[y, y\right]p_x$

$$=x\left[y, p_y\right]+0+0+0=i\hbar x$$

where we have used $\left[y, p_y\right]=i\hbar x,\ \left[y, x\right]=\left[y, p_x\right]=\left[y, y\right]=0$

Q31. A particle of mass m is in a one dimensional infinite potential well of length L, extending from $x=0$ to $x=L$. When it is in the energy Eigen-state labelled by $n,(n=1,2,3,..)$ the probability of finding in the interval $0\le x\le L/8$ is 1/8. The minimum value of n for which this is possible is

(a) 4 (b) 2 (c) 6 (d) 8

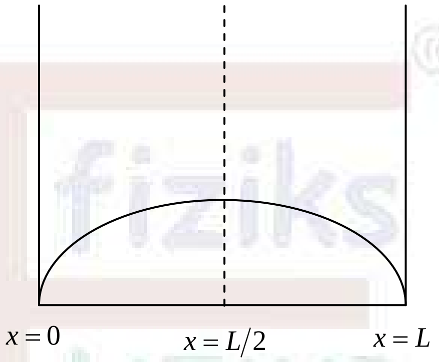
Ans. 31: (a)

Solution: This problem is solved using the wavefunction.

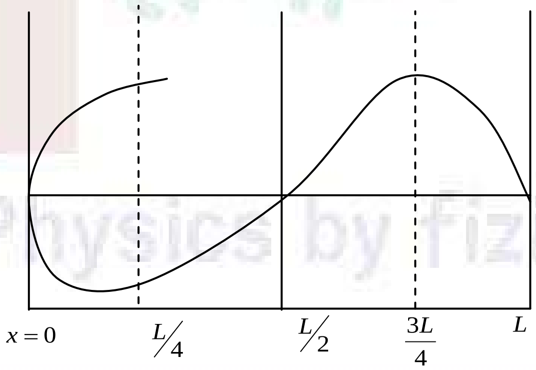
(a) The plot for $\psi_1(x)$ between $0<x<L$ is

The probability of finding the particle in region $0<x<L/2$ and $\frac{L}{2}<x<L$ is

$$P\left(0<x<L/2\right)=P\left(\frac{L}{2}<x<L\right)=\frac{1}{2}$$



(b) The plot for $\psi_2(x)$ in between $0<x<L$



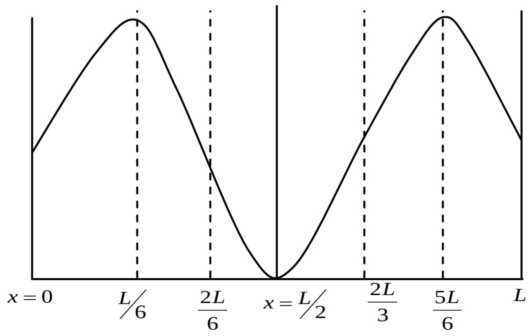
The probability of finding the particle in region.

$0<x<L/4$;

$\frac{L}{4}<x<\frac{L}{2}$; $\frac{L}{2}<x<\frac{3L}{4}$; $\frac{3L}{4}<x<L$ is

$$P\left(0<x<\frac{L}{4}\right)=P\left(\frac{L}{4}<x<\frac{L}{2}\right)=P\left(\frac{L}{2}<x<\frac{3L}{4}\right)=P\left(\frac{3L}{4}<x<L\right)=\frac{1}{4}$$

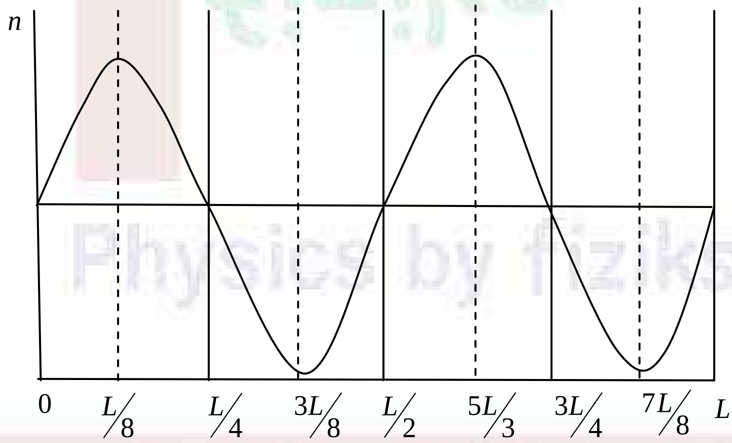
(c) The plot for $\psi_3(x)$ in region $0<x<L$ is



The plot is divided in 6 equal region of $0<x<L/6$; $\frac{L}{6}<x<2L/6$ $\frac{2L}{6}<x<\frac{L}{2}$; $\frac{L}{2}<x<\frac{2L}{3}$; $\frac{2L}{3}<x<\frac{5L}{6}$; $\frac{5L}{6}<x<L$.

The probability of finding the particle in each of region is $1/6$.

(d) The plot for $\psi_4(x)$ in region $0<x<L$ is



The wave function is divided in 8 equal region of $0<x<\frac{L}{8}, \frac{L}{8}<x<\frac{L}{4}$, $\frac{L}{4}<x<\frac{3L}{8}$, $\frac{3L}{8}<x<\frac{L}{2}$, $\frac{L}{2}<x<\frac{5L}{8}$, $\frac{5L}{8}<x<\frac{3L}{4}$, $\frac{3L}{4}<x<\frac{7L}{8}$, $\frac{7L}{8}<x<L$.

The probability of finding the particle in each of these region is $1/8$.

Thus, the value of $n=4$, such that the probability of finding the particle in region $P\left(0<x<L/8\right)=1/8$.

Q38. A two-state system evolves under the action of the Hamiltonian $H = E_0|A\rangle\langle A| + (E_0+\Delta)|B\rangle\langle B|$, where $|A\rangle$ and $|B\rangle$ are its two orthonormal states. These states transform to one another under parity, i.e. $P|A\rangle = |B\rangle$ and $P|B\rangle = |A\rangle$. If at time $t=0$ the system is in a state of definite parity $P=1$, the earliest time t at which the probability of finding the system in a state of parity $P=-1$ is one is

(a) $\frac{\pi\hbar}{2\Delta}$ (b) $\frac{\pi\hbar}{\Delta}$ (c) $\frac{3\pi\hbar}{2\Delta}$ (d) $\frac{2\pi\hbar}{\Delta}$

Ans. 38: (b)

Solution.: The Hamiltonian for the two state system is given by $H = \varepsilon_0|A\rangle\langle A| + (\varepsilon_0 + D)|B\rangle\langle B|$

In matrix, $H = \begin{pmatrix} E_0 & 0 \\ 0 & \varepsilon_0 + D \end{pmatrix}$

The energy eigenvalue for the system is given by $|H - \lambda I| = \begin{vmatrix} \varepsilon_0 - \lambda & 0 \\ 0 & (\varepsilon_0 + 0) - \lambda \end{vmatrix} = 0$

or $\lambda = \varepsilon_0,\ \varepsilon_0 + 0$

The eigenfunction of the system is given by $|\phi(t)\rangle = |A\rangle e^{-\frac{iE_0}{\hbar}t} + |B\rangle e^{\frac{-i(\varepsilon_0+0)}{\hbar}t}$

According to question, we have $\pi|\phi(t)\rangle = -|\phi(t)\rangle$

$$\pi|A\rangle e^{-i\varepsilon_0 t/\hbar} + \pi|B\rangle e^{\frac{-i(\varepsilon_0+D)}{\hbar}t} = -|A\rangle e^{-i\varepsilon_0 t/\hbar} - |B\rangle e^{\frac{-(\varepsilon_0+D)}{\hbar}t}$$

$$|B\rangle e^{\frac{-i\varepsilon_0 t}{\hbar}} + |A\rangle e^{\frac{-i(\varepsilon_0+D)t}{\hbar}} = -|A\rangle e^{-i\varepsilon_0 t/\hbar} - |B\rangle e^{\frac{-i(E_0+D)}{\hbar}t}$$

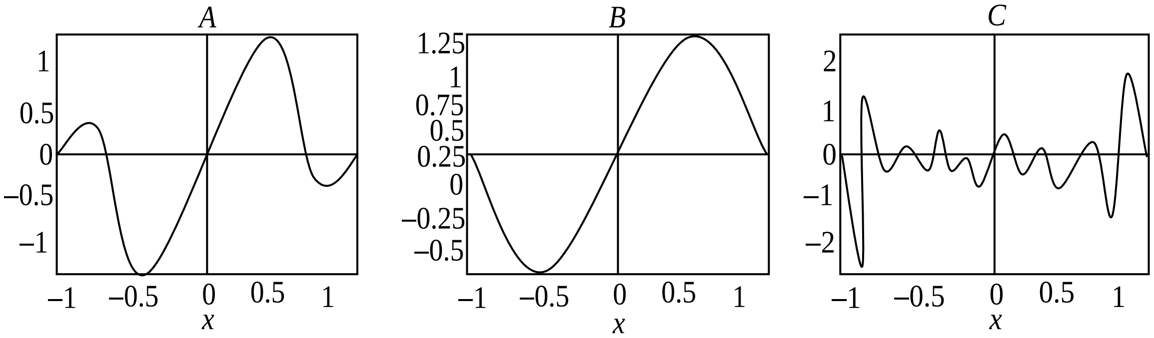
Comparing coefficient of state $|A\rangle$ and $|B\rangle$, we get

For A: $-e^{-i\varepsilon_0 t/\hbar} = e^{\frac{-i(\varepsilon_0+D)t}{\hbar}}$; For B: $-e^{\frac{-i(E_0+D)t}{\hbar}} = e^{-i\varepsilon_0 t/\hbar}$

Since both these conditions are same, we $-e^{-i\varepsilon_0 t/\hbar} = e^{-i(\varepsilon_0+0)t/\hbar}$ or $e^{-i(\varepsilon_0+0)t/\hbar}e^{-i\varepsilon_0 t/\hbar} - 1$

$e^{\frac{i\Delta t}{\hbar}} = e^{i\pi} \Rightarrow i\frac{\Delta t}{\hbar} = i\pi$ or $t = \frac{\hbar\pi}{\Delta}$ Thus the correct option is (b)

Q42. The figures below depict three different wave functions of a particle confined to a one dimensional box $-1 \le x \le 1$



The wave functions that correspond to the maximum expectation values $|\langle x \rangle|$ (absolute value of the mean position) and $\langle x^2 \rangle$, respectively, are

(a) B and C (b) B and A (c) C and B (d) A and B

Ans. 42: (a)

Solution: This problem is solved using properties:

(1) For a box of length $-a < x < d$, $\langle x \rangle$ is always zero.

(2) For a $60x$ of length $-a < x < d$, $|\langle x \rangle|$ is always non zero.

(3) The wavefunction is of the form $\psi(x) = A\left(a^2 - x^2\right) \qquad x = \pm a$

The normalised wave function is given by $\psi(x) = \sqrt{\frac{15}{16a^3}}\left(a^2 - x^2\right)$

The expectation value of $\langle x^2 \rangle$ is $\langle x^2 \rangle = \frac{15}{16a^5}\int_{-a}^{d} x^2\left(a^2 - x^2\right)d\,x = \frac{16a^7}{105}$

Thus, at $a = \pm 1$ curve would take maximum and minimum values.

For $|\langle x \rangle|$ the curve given in the option (b) is non-zero.

For $\langle x^2 \rangle$, the curve takes maximum and minimum value at $a = \pm 1$ in the curve shown in option (c).

Q43. The Hamiltonian of a particle of mass m in one-dimension is $H = \frac{1}{2m}p^2 + \lambda |x|^3$, where $\lambda > 0$ is a constant. If E_1 and E_2 respectively, denote the ground state energies of the particle for $\lambda = 1$ and $\lambda = 2$ (in appropriate units) the ratio E_2 / E_1 is best approximated by

(a) 1.260 (b) 1.414 (c) 1.516 (d) 1.320

Ans. 43: (d)

Solution: Consider the potential of the particle of form $V(x)=\lambda|x|^n$

The ground state energy of the particle using with approximation depends on λ or $\varepsilon_n \alpha \lambda^{\frac{2}{\lambda+2}}$

So the ration $\varepsilon_2 | \varepsilon_1$ is given by $\frac{\varepsilon_2}{\varepsilon_1}=\left(\frac{\lambda_2}{\lambda_1}\right)^{\frac{2}{m+2}}=\left(\frac{2}{1}\right)^{\frac{2}{3+2}}=2^{2/5}$ or $\frac{\varepsilon_2}{\varepsilon_1}=1.319\approx 1.32$.

Part C

Q49. The energies of a two-state quantum system are E_0 and $E_0+\alpha\hbar$, (where $\alpha>0$ is a constant) and the corresponding normalized state vectors are $|0\rangle$ and $|1\rangle$, respectively. At time $t=0$, when the system is in the state $|0\rangle$, the potential is altered by a time independent term V such that $\langle 1|V|0\rangle=\hbar\alpha/10$. The transition probability to the state $|1\rangle$ at times $t\ll 1/\alpha$, is

(a) $\alpha^2t^2/25$ (b) $\alpha^2t^2/50$ (c) $\alpha^2t^2/100$ (d) $\alpha^2t^2/200$

Ans. 49: (c)

Solution: The transmission probability to the state $|1\rangle$ at time t is

$$P_{0\to 1}=\frac{1}{\hbar^2}\left|\langle 1|\forall|0\rangle\right|^2 \Big/ \int_0^t e^{i\left(\frac{\varepsilon_1-\varepsilon_0}{\hbar}\right)t}dt$$

$$=\frac{1}{\hbar^2}\left(\frac{\hbar\alpha}{10}\right)^2\left|\int_0^t e^{i}\frac{\alpha\hbar}{\hbar}t\,dt\right|^2=\frac{\alpha^2}{100}\left|\int_0^t e^{i\alpha t}dt\right|^2=\frac{\alpha^2}{100}\left|\int_0^t dt\right|^2=\frac{\alpha^2t^2}{100}$$

where, we have used $e^{i\alpha t}\approx 1$ as $\alpha t \lll 1$

Q61. A particle of mass m in one dimension is in the ground state of a simple harmonic oscillator described by a Hamiltonian $H=\frac{p^2}{2m}+\frac{1}{2}m\omega^2x^2$ in the standard notation. An impulsive force at time to $t=0$ suddenly imparts a momentum $p_0=\sqrt{\hbar m\omega}$ to it. The probability that the particle remains in the original ground state is

(a) e^{-2} (b) $e^{-3/2}$ (c) e^{-1} (d) $e^{-1/2}$

Ans. 61: (d)

Solution: The new state of the system is $\psi_{P_0}(x)=e^{-ip_0x/\hbar}\psi_0(x)=\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}e^{-ip_0x\hbar}e^{-m\omega x^2/2\hbar}$

In an expansion in the complete set of harmonic oscillator eigenfunction.

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$$\psi_{P_0}(x)=\sum_{n=0}^{\infty}C_n\psi_n(x)$$

the coefficient $C_n=\int_{-\infty}^{\infty}dx\,\psi_n^*(x)\psi_{P_0}(x)$

are the probability amplitudes for the system in the state ψ_n. Thus

$$P_0=\left|\int\psi_0(x)\psi_{P_0}(x)dx\right|^2=\left|\int\psi_0^2(x)e^{-|p_0x|\hbar}dx\right|^2$$

Calculating the Guarian integral $\int_{-\infty}^{\infty}dx\,e^{\left(-\frac{i}{\hbar}xp_0-\frac{m\omega}{\hbar}x^2\right)}=\sqrt{\frac{\pi}{t}}e^{-\frac{(P_0/\hbar)^2}{4(m\omega/\hbar)}}=\sqrt{\frac{\pi\hbar}{m\omega}}e^{-\frac{p_0^2}{4m\omega\hbar}}$

Substituting value in expression of probability.

$$P_0\left(\frac{m\omega}{\pi\hbar}\right)^{1/2}\left|\int_{-\infty}^{\infty}e^{\left(-\frac{i}{\hbar}p_0x-\frac{m\omega}{\hbar}x^2\right)}dx\right|^2=\left|\left(\frac{m\omega}{\pi\hbar}\right)^{1/2}\left(\frac{\pi\hbar}{m\omega}\right)^{1/2}e^{-\frac{p_0^2}{4m\omega\hbar}}\right|^2=e^{-\frac{p_0^2}{2m\omega\hbar}}$$

We get $P_0=e^{-\frac{p_0^2}{2m\omega\hbar}}=e^{-\frac{\left(\sqrt{m\omega\hbar}\right)^2}{2m\omega\hbar}}=e^{-1/2}$

where, we have used $P_0=\sqrt{m\omega\hbar};\ \int_{-\infty}^{\infty}e^{-i\alpha x-\beta x^2}dx=\sqrt{\frac{\pi}{\beta}}e^{-\frac{\alpha^2}{4\beta}}$

Q64. The $|3,0,0\rangle$ state in the standard notation $|n,l,m\rangle$ of the H-atom in the non-relativistic theory decays to the state $|1,0,0\rangle$ via two dipole transition. The transition route and the corresponding probability are

(a) $|3,0,0\rangle\to|2,1,-1\rangle\to|1,0,0\rangle$ and $1/4$ (b) $|3,0,0\rangle\to|2,1,1\rangle\to|1,0,0\rangle$ and $1/4$

(c) $|3,0,0\rangle\to|2,1,0\rangle\to|1,0,0\rangle$ and $1/3$ (d) $|3,0,0\rangle\to|2,1,0\rangle\to|1,0,0\rangle$ and $2/3$

Ans. 64: (c)

Solution: For, dipole transition,

$\Delta l=\pm1 \quad and \quad \Delta m=0,\pm1$

For all options, n = 2, so l = 0, 1

For l = 0, m = 0 and for l = 1, m = -1, 0, 1

The transitions $|3,0,0\rangle\longrightarrow|1,0,0\rangle$ via $|2,1,m\rangle$ for m = -1, 0, 1 are all valid according to the dipole transition rule. Thus, there are three different states through which the $|3,0,0\rangle$ state can decay to $|1,0,0\rangle$ each with equal probability. Hence each transition has a probability of 1/3. So, option (c) with probability 1/3 is correct.

Q74. In an elastic scattering process at an energy E, the phase shifts satisfy $\delta_0 \approx 30^{\circ}, \delta_1 \approx 10^{\circ}$, while the other phase shifts are zero. The polar angle at which the differential cross section peaks is closest to

(a) 20° (b) 10° (c) 0° (d) 30°

Ans. 74: (c)

Solution: In the partial wave expansion, the differential scattering cross section is given by

$$\frac{d\theta}{d(\cos\theta)} = \left|\sum_{\ell}(2\ell+1)e^{i\delta e}\sin\delta e\psi_{\ell}(\cos\theta)\right|^2$$

where θ is the scattering angle. Taking 'Cross section for $\ell = 0$ and $\ell = 1$, we have,

$$\frac{d\theta}{d(\cos\theta)} = \left|e^{i\delta\theta}\sin\delta\theta + 3e^{i\delta_1}\sin\delta_1\cos\theta\right| = 0$$

Since the differential cross section peaks is do not,

$$\frac{d\theta}{d(\cos\theta)} = \left|e^{i\delta\theta}\sin\delta\theta + 3e^{i\delta_1}\sin\delta_1\cos\theta\right| = 0 \quad \text{or} \quad \cos\theta = \frac{-e^{i\delta\theta}\sin\delta\theta}{3e^{i\delta_1}\sin\delta_1}$$

Simplifying above expression

$$\cos\theta = \frac{-1}{3}\frac{2\cos\delta\theta\sin\delta\theta}{2\cos\delta_1\sin\delta_1} = -\frac{1}{3}\frac{\sin 2\delta\theta}{\sin 2\delta_1} \Rightarrow \cos\theta = -\frac{1}{3}\frac{\sin 2\times 30}{\sin 2\times 10} = -\frac{1}{3}\frac{\sin 60}{\sin 20}$$

$$\Rightarrow \cos\theta = -\frac{1}{3}\times\frac{0.8660}{0.342} = -0.844 \quad \Rightarrow \cos\theta = 0.844 \Rightarrow \theta = 32°.4$$

Thus, the closest angle would be 30°.

Q75. The unnormalized wave function of a particle in one dimension in an infinite square well with walls at $x = 0$ and $x = a$, is $\psi(x) = x(a-x)$. If $\psi(x)$ is expanded as a linear combination of the energy eigenfunctions, $\int_0^a |\psi(x)|^2 dx$ is proportional to the infinite series

(You may use $\int_0^a t\sin t\,dt = -a\cos a + \sin a$ and $\int_0^a t^2\sin t\,dt = -2-(a^2-2)\cos a + 2a\sin a$

(a) $\sum_{n=1}^{\infty}(2n-1)^{-6}$ (b) $\sum_{n=1}^{\infty}(2n-1)^{-4}$ (c) $\sum_{n=1}^{\infty}(2n-1)^{-2}$ (d) $\sum_{n=1}^{\infty}(2n-1)^{-8}$

Ans. 75: (a)

Solution: We have, $\psi(x_1, t=0) = \alpha(a-x)$

The normalization constant is determined at follows.

$$\int|\psi(x)|^2 dx = A^2\int_0^a x^2(a-x)^2 dx = A^2\frac{a^5}{30} = 1 \quad \text{or} \quad A = \sqrt{\frac{a^5}{30}}$$

Thus, the normalised wave function is given by $\psi(x_1, t=0) = x(a-x)\cdot\sqrt{\frac{30}{a^5}}$

We expand $\psi(x,t=0)$ in terms of energy eigen state $\psi(x,0) = \sum_{n=1}^{\infty} C_n \psi_n(x)$.

Multiply the above equation by $\psi_n^*(x)$ and integrate to determine coefficient C_n,

$$C_n = \int_0^L \psi_n^*(x)\psi(x,0)\,dx = \left(\frac{30}{a^5}\right)^{1/2}\left(\frac{2}{a}\right)\int_0^a x(a-x)\sin\frac{n\pi x}{d}dx$$

Let change of variable $y = \pi x/L$; $C_n = \frac{2\sqrt{15}}{\pi^2}\int_0^{\pi} y\left(1-\frac{y}{\pi}\right)\sin ny\,dy$

Employing integral $\int_0^{\pi} y \sin ny = -\frac{\pi}{n}(-1)^n$, $\int_0^{\pi} y^2 \sin ny\,dy = -\frac{\pi^2}{n}(-1)^n + \frac{2}{n^3}\left[(-1)^n - 1\right]$

We get $C_n = \frac{4\sqrt{15}}{\pi^3 n^3}\left[1-(-1)^n\right]$. Probability P_n is given by $P_n = |a_n|^2 = \frac{240}{\pi^6 n^6}\left[1-(-1)^n\right]^2$

One can see that P_n is proportional to n^{-6}, this is assessable in option (1). Hence the correct series would by $\sum_{n=1}^{\infty}(2n-1)^{-6}$.

Part B

Q33. The volume and temperature of a spherical cavity filled with black body radiation are V and $300K$, respectively. If it expands adiabatically to a volume $2V$, its temperature will be closest to

(a) $150\,K$ (b) $300\,K$ (c) $250\,K$ (d) $240\,K$

Ans. 33: (d)

Solution:

$V_1 = V$, $T_1 = 300K$, $V_2 = 2V$, $T_2 = ?$

$$VT^3 = \text{constant} \quad \Rightarrow V_1T_1^3 = V_2T_2^3 \quad \Rightarrow T_2^3 = \left(\frac{V_1}{V_2}\right)T_1^3$$

$$\Rightarrow T_2 = \left(\frac{V_1}{V_2}\right)^{1/3} \Rightarrow T_1 = \left(\frac{1}{2}\right)300 = 0.8\times 300 = 240\,K$$

$\therefore$ (d) is correct

Q34. The ratio c_p/c_v of the specific heats at constant pressure and volume of a monatomic ideal gas in two dimensions is

(a) $3/2$ (b) 2 (c) $5/3$ (d) $5/2$

Ans. 34: (b)

Solution:

For monoatomic ideal gas in 2D; $E = \frac{p_x^2}{2m} + \frac{p_y^2}{2m}$

$$\Rightarrow \langle E\rangle = \frac{1}{2}kT + \frac{1}{2}kT = kT$$

$$U = NKT = nN_AkT = nRT \quad \Rightarrow C_V = \left(\frac{dU}{dT}\right)_V = nR$$

$$C_P - C_V = nR \quad \Rightarrow C_P = nR + nR = 2nR \Rightarrow \frac{C_P}{C_V} = \frac{2nR}{nR} = 2$$

$\therefore$ (b) is correct

Q35. The total number of phonon modes in a solid of volume V is $\int_0^{\omega_D} g(\omega)d\omega = 3N$, is the number of primitive cells, ω_D is the Debye frequency and density of photon modes is $g(\omega) = AV\omega^2$ (with $A > 0$ a constant). If the density of the solid doubles in a phase transition, the Debye temperature θ_D will

(a) increase by a factor of $2^{2/3}$ (b) increase by a factor of $2^{1/3}$

(c) decrease by a factor of $2^{2/3}$ (d) decrease by a factor of $2^{1/3}$

Ans. 35: (b)

Solution:

Deby temperature is $Q_D = \left(\frac{hv_s}{k_B}\right)\left(6\pi^2 \frac{N}{V}\right)^{1/3} = \left(\frac{hv_s}{k_B}\right)\left(6\pi^2\delta\right)^{1/3} \Rightarrow Q_D = A\delta^{1/3}$

Now, if density doubles $\delta' = 2\delta$;

$Q_B' = A(\delta')^{1/3} = A(2\delta)^{1/3} = A\delta^{1/3} \cdot 2^{1/3}$

$\therefore \boxed{Q_D' = 2^{1/3}Q_D}$

Thus Q_D increases by a factor of $2^{1/3}$.

Part C

Q53. The dispersion relation of a gas of non-interacting bosons in d dimensions $E(k) = ak^s$ where a and s are positive constants, Bose-Einstein condensation will occur for all values of

(a) $d > s$ (b) $d + 2 > s > d - 2$

(c) $s > 2$ independent of d (d) $d > 2$ independent of s

Ans. 53: (a)

Solution: Given the dispersion relation $E(k) = ak^s$ .....(1)

For a non-relativistic system in 3-D; $E = \frac{p^2}{2m} \Rightarrow \frac{\hbar^2k^2}{2m} \Rightarrow s = 2,\ d = 3 \quad \Rightarrow s < d$

Similarly in relativistic case in 3-D; $E = pc = \hbar Ck, \ \Rightarrow s = 1,\ d = 3 \ \Rightarrow s < d$

In 2-D relativistic $E = pC, \quad p = \sqrt{p_x^2 + py^2}, \quad s = 1,\ d = 2 \quad s < d$

$\therefore$ (a) is correct

Q56. The energy levels of a non-degenerate quantum system are $\in_n = nE_0$, where E_0 is a constant and $n = 1,2,3,\ldots$. At a temperature T, the free energy F can be expressed in terms of the average energy E by

(a) $E_0 + k_B T\ln\frac{E}{E_0}$ (b) $E_0 + 2k_B T\ln\frac{E}{E_0}$ (c) $E_0 - k_B T\ln\frac{E}{E_0}$ (d) $E_0 - 2k_B T\ln\frac{E}{E_0}$

Ans. 56: (c)

Solution: $E_n \to nE_0$; $\quad E_3 \to 3E_0,\ E_2 \to 2E_0,\ E_1 \to E_0$

$$Q = e^{-\beta E_0} + e^{-2\beta E_0} + e^{-3\beta E_0} = e^{-\beta E_0}\left[1 + e^{-\beta E_0} + e^{-2\beta E_0} + \ldots\right] = e^{-\beta E_0} \times \frac{1}{1-e^{-\beta E_0}} = \frac{1}{e^{\beta E_0}-1} \qquad \ldots(1)$$

Now Helmholtz free energy $F = -k_B T\ln Q = k_B T\ln\left(e^{\beta E_0}-1\right)$ ....(2)

$$\text{Now } \langle E\rangle = -\frac{\partial \ln}{\partial\beta} = \frac{\partial}{\partial\beta}\ln\left(e^{\beta E}-1\right) = \frac{e^{\beta E_0}\left(E_0\right)}{\left(e^{\beta E_0}-1\right)} \quad \text{i.e., } \frac{E_0}{\langle E\rangle}e^{\beta E_0} = \left(e^{\beta E_0}-1\right) \qquad \ldots(3)$$

$$\text{From (2) \& (3); } F = k_B T\ln\left(\frac{E_0}{E}e^{\beta E_0}\right) = k_B T\left[\ln\left(\frac{E_0}{E}\right) + \ln e^{\beta E_0}\right] = k_B T\left[\ln\left(\frac{E_0}{E}\right) + \frac{E_0}{k_B T}\right]$$

$$\boxed{F = E_0 - k_B T\ln\left(\frac{E}{E_0}\right)}$$

$\therefore$ (c) is correct.

Q62. A polymer, made up of N monomers, is in thermal equilibrium at temperature T. Each monomer could be of length a or $2a$. The first contributes zero energy, while the second one contributes $\in$. The average length (in units of Na) of the polymer at temperature $T = \in / k_B$ is

(a) $\frac{5+e}{4+e}$ (b) $\frac{4+e}{3+e}$ (c) $\frac{3+e}{2+e}$ (d) $\frac{2+e}{1+e}$

Ans. 62: (d)

Solution: When length of monomer is a, energy = 0; $\quad 2a$, energy = ε

$$\text{Now } P(\varepsilon) = \frac{g_i e^{-\beta\varepsilon_i}}{\sum g_i e^{-\beta\varepsilon_1}}, \text{ Here } g_i = 1 \quad \therefore P(\varepsilon = 0) = \frac{e^{-\beta\cdot 0}}{e^{-\beta\cdot 0} + e^{-\beta\varepsilon}} = \frac{1}{1+e^{-\beta\varepsilon}}, \quad P(\varepsilon = \varepsilon) = \frac{e^{-\beta\varepsilon}}{1+e^{-\beta\varepsilon}}$$

$$\langle L\rangle = NaP(0) + 2NaP(\varepsilon) = \frac{Na}{1+e^{-\beta\varepsilon}} + \frac{2Na\,e^{-\beta\varepsilon}}{1+e^{-\beta\varepsilon}}$$

$$\text{Let } T = \varepsilon/k_B \Rightarrow \beta = \frac{1}{k_0 T} = \frac{1}{k_B \times \varepsilon/k_B} \Rightarrow \beta\varepsilon = 1$$

$$\therefore \langle L\rangle = Na\left[\frac{1+2e^{-\beta\varepsilon}}{1+e^{-\beta\varepsilon}}\right] = Na\left[\frac{1+2e^{-1}}{1+e^{-1}}\right] \qquad \Rightarrow \boxed{\langle L\rangle = Na\left[\frac{e+2}{e+1}\right]}$$

$\therefore$ (d) is correct .

Q65. Balls of ten different colours labeled by $a = 1, 2, ..., 10$ are to be distributed among different coloured boxes. A ball can only go in a box of the same colour, and each box can contain at most one ball. Let n_a and N_a denote respectively, the numbers of balls and boxes of colour a. Assuming that $N_a >> n_a >> 1$, the total entropy (in units of the Boltzmann constant) can be best approximated by

(a) $\sum_a \left(N_a \ln N_a + n_a \ln n_a - (N_a - n_a)\ln(N_a - n_a)\right)$

(b) $\sum_a \left(N_a \ln N_a - n_a \ln n_a + (N_a - n_a)\ln(N_a - n_a)\right)$

(c) $\sum_a \left(N_a \ln N_a - n_a \ln n_a + (N_a - n_a)\ln(N_a - n_a)\right)$

(d) $\sum_a \left(N_a \ln N_a + n_a \ln n_a + (N_a - n_a)\ln(N_a - n_a)\right)$

Ans. 65: (b)

Solution: Let n_1 balls of colour 1 to be distributed in N_1 boxes of colour 1

n_2 balls of colour 2 to be distributed in N_2 boxes of colour 2

n_{10} balls of colour 10 to be distributed in N_{10} boxes of colour 10

$$\therefore \Omega_{Total} = \frac{N_1!}{n_1!(N_1 - n_1)!} + \frac{N_2!}{n_2!(N_2 - n_2)!} + ... + \frac{N_{10}!}{n_{10}!(N_{10} - n_{10})!} = \sum_a \frac{Na!}{n_a!(N_a - n_a)!}$$

$$S = k_B \ln \Omega = k_B \sum_a \ln\left[\frac{Na!}{n_a!(N_a - n_a)!}\right]$$

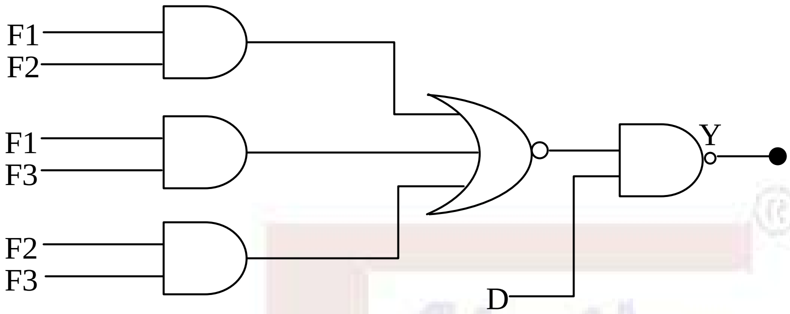
$$\frac{S}{k_B} = \sum_a \left(N_a \ln Na - n_a \ln n_a - (N_a - n_a)\ln(N_a - n_a)\right)$$

$\therefore$ (b) is correct.

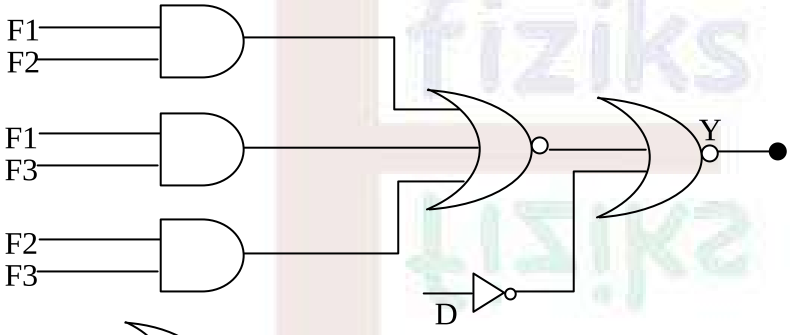
Part B

Q27. The door of an X -ray machine room is fitted with a sensor D (0 is open and 1 is closed). It is also equipped with three fire sensors F_1, F_2 and F_3 (each is 0 when disabled and 1 when enabled). The X -ray machine can operate only if the door is closed and at least 2 fire sensors are enabled. The logic circuit to ensure that the machine can be operated is

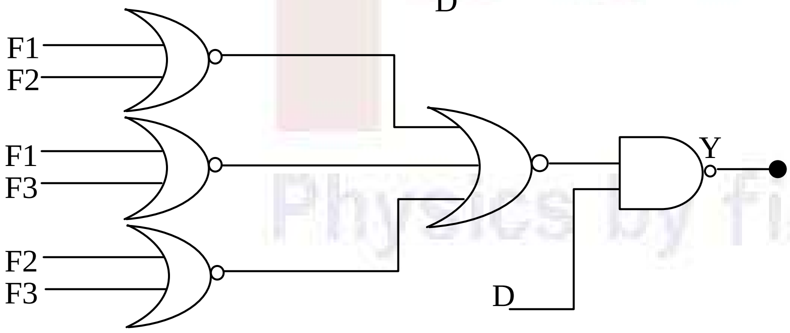
(a)



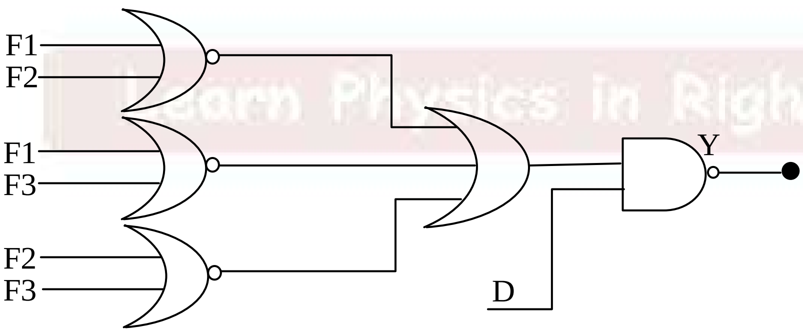
(b)



(c)



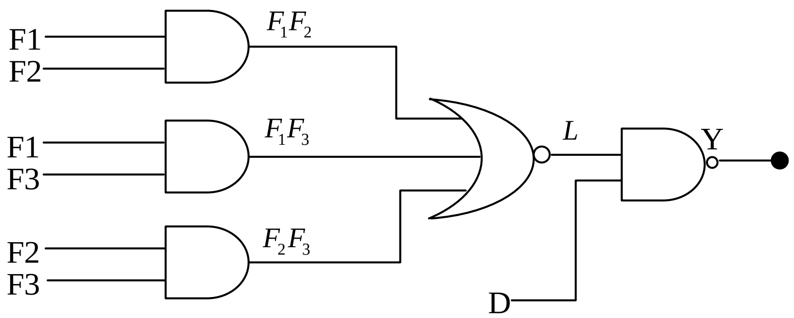
(d)



Ans. 27: option (a), (b) and (d) are possible

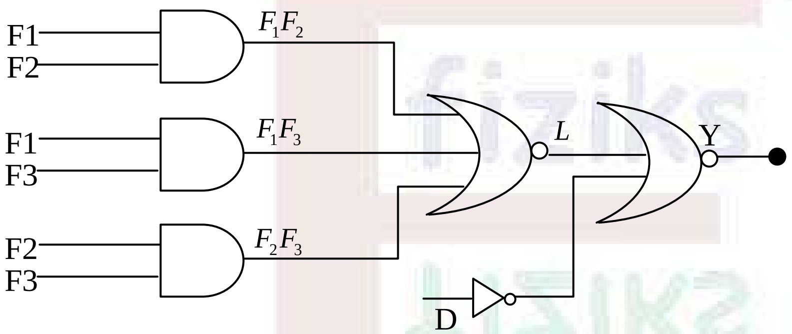
Solution.:

(a)



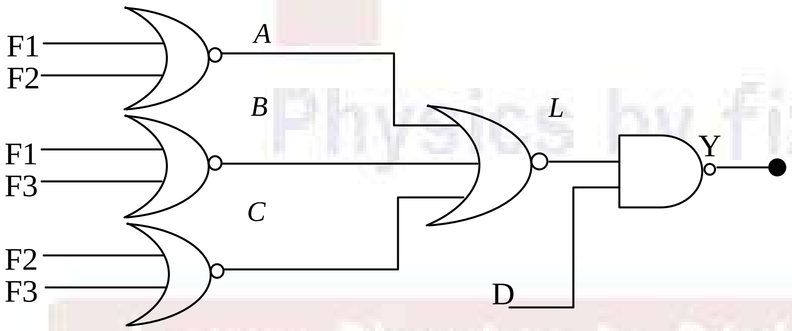
$L=\overline{F_1F_2+F_1F_3+F_2F_3}$, $D=1$, $Y=\overline{L.D}=\overline{L}+\overline{D}=\overline{\overline{F_1F_2+F_1F_3+F_2F_3}}+\overline{1}=F_1F_2+F_1F_3+F_2F_3$

(b)



$L=\overline{F_1F_2+F_1F_3+F_2F_3}$, $D=1$, $Y=\overline{L+\overline{D}}=\overline{L}\,\overline{\overline{D}}=\left(\overline{\overline{F_1F_2+F_1F_3+F_2F_3}}\right)1=F_1F_2+F_1F_3+F_2F_3$

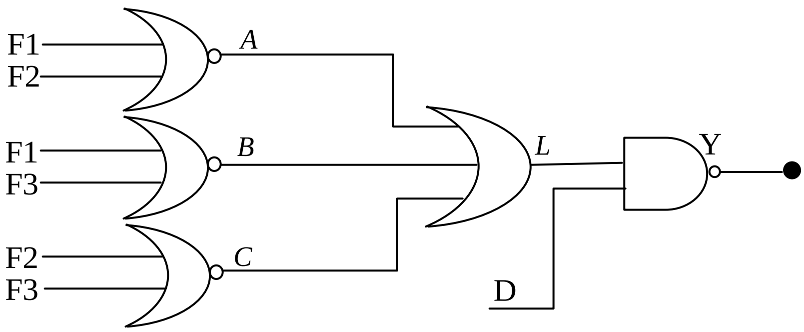
(c)



$L=\overline{A+B+C}$, $D=1$, $Y=\overline{L.D}=\overline{L}+\overline{D}=\left(\overline{\overline{A+B+C}}\right)+0=A+B+C$

$Y=\left(\overline{F_1+F_2}\right)+\left(\overline{F_1+F_3}\right)+\left(\overline{F_2+F_3}\right)=\overline{F_1}\,\overline{F_2}+\overline{F_1}\,\overline{F_3}+\overline{F_2}\,\overline{F_3}$

(d)

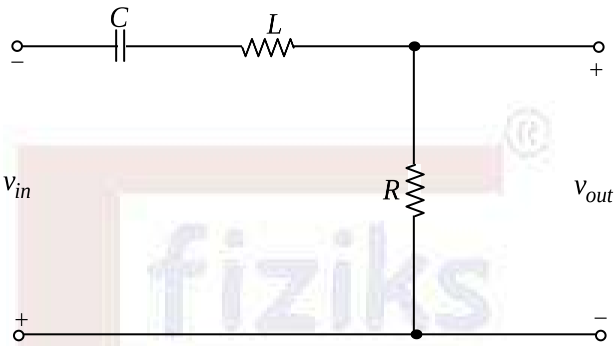


$L = A + B + C$, $D = 1$ $Y = \overline{L.D} = \overline{L} + \overline{D} = \left(\overline{A+B+C}\right) + 0 = \overline{A+B+C}$

$$Y = \overline{\left(\overline{F_1+F_2}\right)+\left(\overline{F_1+F_3}\right)+\left(\overline{F_2+F_3}\right)} = \left(\overline{\overline{F_1+F_2}}\right)\left(\overline{\overline{F_1+F_3}}\right)\left(\overline{\overline{F_2+F_3}}\right)$$

$$\Rightarrow Y = \left(F_1+F_2\right)\left(F_1+F_3\right)\left(F_2+F_3\right)$$

Q28. In the LCR circuit shown below, the resistance $R = 0.05\,\Omega$, the inductance $L = 1H$ and the capacitance $C = 0.04\,F$.



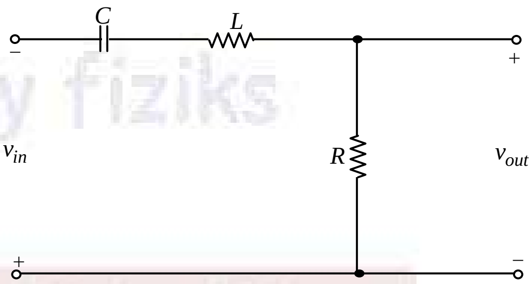
If the input v_{in} is a square wave of angular frequency $1\,rad/s$, the output v_{out} is best approximated by a

(a) Square wave of angular frequency $1\,rad/s$ (b) Sine wave of angular frequency $1\,rad/s$

(c) Square wave of angular frequency $5\,rad/s$ (d) Sine wave of angular frequency $5\,rad/s$

Ans. 28: (d)

Solution.: $v_{in} = 1\,rad/s, L = 1H, C = 0.04F$



Resonant angular frequency

$$\omega_r = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{0.04}} = 5\,rad/s$$

Thus, for an input frequency of $1\,rad/s$ (just like dc), the LC-circuit will oscillate in sinusoidal fashion (it can only oscillate harmonically), at $5\,rad/s$. Hence, (d) is the correct answer.

Q32. In an experiment, the velocity of a non-relativistic neutron is determined by measuring the time $(\sim 50\,ns)$ it takes to travel from the source to the detector kept at a distance L. Assume that the error in the measurement of L is negligibly small. If we want to estimate the kinetic energy T of the neutron to within 5% accuracy, i.e., $|\delta T/T| \le 0.05$, the maximum permissible error $|\delta T|$ in measuring the time of flight is nearest to

(a) $1.75\,ns$ (b) $0.75\,ns$ (c) $2.25\,ns$ (d) $1.25\,ns$

Ans. 32: (d)

Solution:

If v is the velocity of non-relativistic neutron and t is the time taken to travel distance L

$$\therefore v=\frac{L}{t}$$

Kinetic energy $T=\frac{1}{2}mv^2=\frac{1}{2}m\frac{L^2}{t^2}$

Percentage error is T is $\frac{\delta T}{T}=2\frac{\delta L}{L}+2\frac{\delta t}{t}$

Since $\delta L=0, \frac{\delta T}{T}=0.09$ [maximum permissible error]

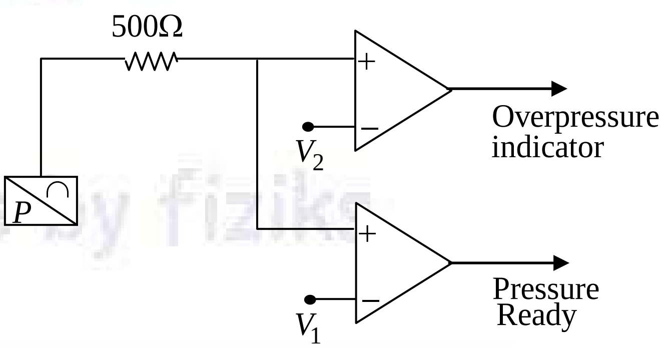
$$\therefore \frac{\delta t}{t}=\frac{1}{2}\frac{\delta T}{T}=\frac{1}{2}\times 0.05=0.025$$

Thus $\delta t=0.025\times t=0.025\times 50$ nsec $\Rightarrow \delta t=1.25$ nsec

Thus, the correct option is (d)

Part C

Q55. The pressure of a gas in a vessel needs be maintained between 1.5 bar to 2.5 bar in an experiment. The vessel is fitted with a pressure transducer that generates $4\,mA$ to $20\,mA$ current for pressure in the range 1 bar to 5 bar. The current output of the transducer has a linear dependence on the pressure.



The reference voltages V_1 and V_2 in the comparators in the circuit (shown in figure above) suitable for the desired operating conditions are respectively

(a) $2V$ and $10V$ (b) $2V$ and $5V$ (c) $3V$ and $10V$ (d) $3V$ and $5V$

Ans. 55: (d)

Solution:

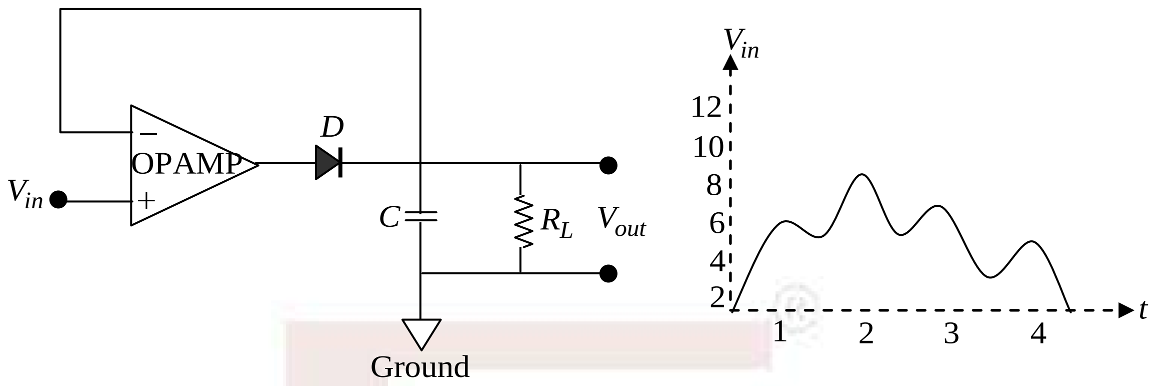
$4\,mA$ to $20\,mA$ current for pressure in the range 1 bar to 5 bar.

So 1 bar corresponds to $4\,mA$.

So 1.5 bar $=6\ mA \Rightarrow V_1=6\ mA\times 500=3.0\ V$

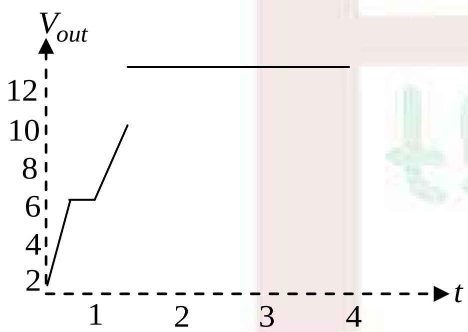
and 2.5 bar $=10\ mA \Rightarrow V_2=10\ mA\times 500=5.0\ V$

Q69. In the following circuit the input voltage V_{in} is such that $|V_{in}| < |V_{sat}|$ where V_{sat} is the saturation voltage of the op-amp (Assume that the diode is an ideal one and $R_L C$ is much larger than the duration of the measurement.)

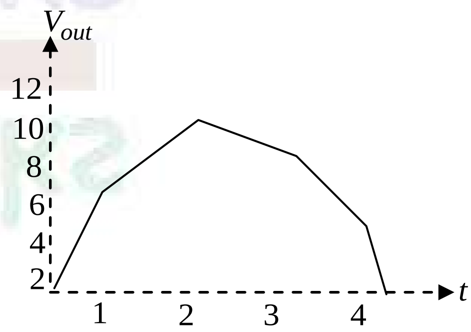


For the input voltage as shown in the figure above the output voltage V_{out} is best represented by

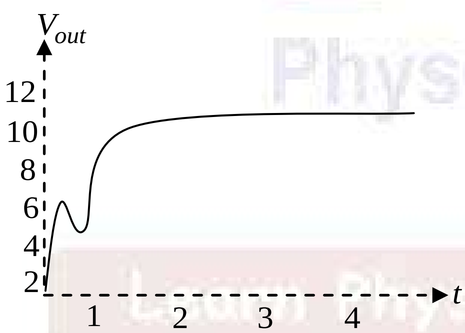
(a)



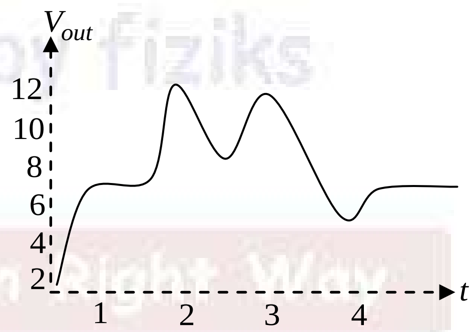
(b)



(c)



(d)



Ans. 69: (a)

Solution: It's a peak detector circuit so options (a) is correct.

Part C

Q52. Diffuse hydrogen gas within a galaxy may be assumed to follow a Maxwell distribution at temperature $10^6\,K$, while the temperature appropriate for the H gas in the inter-galactic space, following the same distribution, may be taken to be $10^4\,K$. The ratio of thermal broadening $\Delta\nu_G/\Delta\nu_{IG}$ of the Lyman-α line from the H-atoms within the galaxy to that from the inter-galactic space is closest to

(a) 100 (b) $1/100$ (c) 10 (d) $1/10$

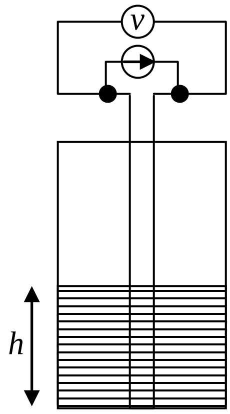
Ans. 52: (c)

Solution: The thermal broadening (or Doppler broadening) is given by $\Delta\nu_D = 1.67\frac{\nu_0}{c}\sqrt{\frac{2kT}{m}}$

Thus $\frac{\Delta\nu_G}{\Delta\nu_{IG}} = \sqrt{\frac{T_G}{T_{IG}}} = \sqrt{\frac{10^6}{10^4}} = 10$

Part C

Q51. To measure the height h of a column of liquid helium in a container, a constant current I is sent through an $NbTi$ wire of length l, as shown in the figure. The normal state resistance of the $NbTi$ wire is R. If the superconducting transition temperature of $NbTi$ is $\approx 10\,K$, then the measured voltage $V(h)$ is best described by the expression



(a) $IR\left(\frac{1}{2}-\frac{2h}{l}\right)$ (b) $IR\left(1-\frac{h}{l}\right)$

(c) $IR\left(\frac{1}{2}-\frac{h}{l}\right)$ (d) $IR\left(1-\frac{2h}{l}\right)$

Ans. 51: (d)

Solution:

Since the superconducting critical temperature for NOT is 30K, the partition of the wire immersed in the liquid Helium is in the superconducting state with zero resistance, while the partition above the liquid is in normal state with resistance R where $R=\frac{\delta l}{A}$

The resistance of the wire of length l is

$$R'=\frac{\delta(l-2h)}{A}\times\frac{l}{l}=\frac{\delta l}{A}\times\frac{l-2h}{l}\Rightarrow R'=R\left(1-\frac{2h}{l}\right)$$

Since, $V=IR' \quad \Rightarrow \boxed{V=IR\left(1-\frac{2h}{l}\right)}$

Thus correct answer is (d).

Q70. Potassium chloride forms an FCC lattice, in which K and Cl occupy alternating sites. The density of KCl is $1.98\,g/cm^3$ and the atomic weights of K and Cl are 39.1 and 35.5, respectively. The angles of incidence (in degrees) for which Bragg peaks will appear when X-ray of wavelength $0.4\,nm$ is shone on a KCl crystal are

(a) $18.5, 39.4$ and 72.2 (b) 19.5 and 41.9

(c) $12.5, 25.7, 40.5$ and 60.0 (d) $13.5, 27.8, 44.5$ and 69.0

Ans. 70: (a)

Solution:

Lattice Parameter is $a^3 = \frac{n_{ca} \times m}{N_A \times \delta} = \frac{4 \times 39.1 + 4 \times 35.5}{6.023 \times 10^{23} \times 1.98} = 2.5 \times 10^{-22}$

$a = 6.3 \times 10^{-8} \text{cm} = 6.3 A^\circ$

Bragg's law is $2d \sin\theta = \lambda \Rightarrow \sin\theta = \frac{\lambda}{2a}\sqrt{h^2 + k^2 + l^2}$

For (200) plane $\sin\theta = \frac{4A^\circ}{2 \times 6.3A^\circ}\sqrt{2^2 + 0 + 0} = \frac{2}{6.3} \times 2$

$\sin\theta = 6.3175 \times 2 = 6.63 \qquad \therefore \theta = \sin^{-1}(0.63) = 39.4^\circ$

Thus option (a) is correct

Q71. Lead is superconducting below $7K$ and has a critical magnetic field 800×10^{-4} tesla close to $0K$. At $2K$ the critical current that flows through a long lead wire of radius $5mm$ is closest to

(a) $1760\,A$ (b) $1670\,A$ (c) $1950\,A$ (d) $1840\,A$

Ans. 71: (d)

Solution: Critical field at temperature T is $B_c(T) = B_c(c)\left[1 - \left(\frac{T}{T_c}\right)^2\right]$

Given $B_c(c) = 800 \times 10^{-4} T,\ T_c = 7K$

$\therefore$ At $T = 2K$, $B_c(2K) = 800 \times 10^{-4}\left[1 - \left(\frac{2}{7}\right)^2\right]$

$B_c(2K) = 800 \times 10^{-4}\left[\frac{49 - 4}{49}\right] = 800 \times 10^{-4}\left(\frac{45}{49}\right)$

Critical current is

$$I_c = \frac{2\pi r B_c(2K)}{\mu_0} = \frac{2\cancel{\pi} \times 5 \times 10^{-3} \times 800 \times 10^{-4}\left(\frac{45}{49}\right)}{4\cancel{\pi} \times 10^{-7}} = 1837A$$

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Part C

Q50. The nuclei of ^{137}Cs decay by the emission of β -particles with a half-life of 30.08 years. The activity (in units of disintegrations per second or Bq) of a $1mg$ source of ^{137}Cs, prepared on January $1, 1980$, as measured on January $1, 2021$ is closest to

(a) 1.79×10^{16} (b) 1.79×10^{9} (c) 1.24×10^{16} (d) 1.24×10^{9}

Ans. 50: (d)

Solution: $A=\lambda N=\lambda N_0 e^{-\lambda t}=\dfrac{0.693}{30.08\,\text{yrs}}\times\dfrac{10^{-3}}{137}\times 6.02\times10^{23}\times e^{-\frac{0.693}{30.08}\times 40}$

$$=0.023\times4.3\times10^{18}\times e^{-0.922}\ \text{(Disintegration per year)}$$

$$=0.023\times4.3\times10^{18}\times0.3977=0.0393\times10^{18}\ \text{(Disintegration per year)}$$

$$=\frac{0.0393\times10^{18}}{365\times24\times60\times60}\ \text{(dps)}=1.24\times10^{9}\ \text{(dps)}$$

Q59. A ^{60}Co nucleus β-decays from its ground state with $J^P=5^+$ to a state of ^{60}Ni with $J^P=4^+$. From the angular momentum selection rules, the allowed values of the orbital angular momentum L and the total spin S of the electron-antineutrino pair are

(a) $L=0$ and $S=1$ (b) $L=1$ and $S=0$ (c) $L=0$ and $S=0$ (d) $L=1$ and $S=1$

Ans. 59: (a)

Solution.: $^{60}Co\rightarrow{}^{60}Ni+\beta^-+\bar{\upsilon}_e$

5^+ 4^+ Here $\Delta J=\pm1,\ \Delta\pi=\text{No}$

So, the given transition is allowed Gamow-Teller transition. So allowed values of the orbital angular momentum L and total spins of the electron-antineutrino pair are $L=0$ and $S=1$

Q72. The Q-value of the α-decay of ^{232}Th to the ground state of ^{228}Ra in $4082\,keV$. The maximum possible kinetic energy of the α-particle is closest to

(a) $4082\,keV$ (b) $4050\,keV$ (c) $4035\,keV$ (d) $4012\,keV$

Ans. 72: (d)

Solution: $Q_\alpha=4082\,KeV$

$$Q_\alpha=\frac{A}{A-4}K_\alpha\Rightarrow 4082=\frac{232}{232-4}\times K_\alpha\ \Rightarrow K_\alpha=\frac{228}{238}\times4082=4012\,KeV$$

Q73. In the reaction $p+n \to p+K^{+}+X$ mediated by strong interaction, the baryon number B, strangeness S and the third component of isospin I_3 of the particle X are, respectively

(a) $-1,-1$ and -1 (b) $+1,-1$ and -1 (c) $+1,-2$ and $-\frac{1}{2}$ (d) $-1,-1$ and 0

Ans. 73: (b)

Solution: $p+n \to p+K^{+}+X$

	p	n	p	K^{+}	X
B	1	1	1	0	1
S	0	0	0	+1	–1
I_3	$+\frac{1}{2}$	$-\frac{1}{2}$	$+\frac{1}{2}$	$+\frac{1}{2}$	–1

Part A

Q1. The arithmetic and geometric means of two numbers are 65 and 25, respectively. What are these two numbers?

(a) $110, 20$ (b) $115,15$ (c) $120,10$ (d) $125,5$

Ans. 1: (d)

Solution:

Suppose two numbers are a and b.

$$AM = \frac{a+b}{2} = 65 \therefore a+b = 130 \qquad \text{(i)}$$

$$GM = \sqrt{ab} = 25 \therefore ab = 625$$

$$(a-b)^2 = (a+b)^2 - 4ab = 16900 - 2500 = 14400$$

$$\therefore a - b = 120 \qquad \text{(ii)}$$

Solving (i) and (ii), we get

$$a = 125, \quad b = 5$$

Q2. An intravenous fluid is given to a child of $7.5\,kg$, at the rate of 20 drop/minute. The prescribed dose of the fluid is 40 ml per kg of body weight. If the volume of a drop is 0.05 ml, how many hours are needed to complete the dose?

(a) 2 (b) 3 (c) 4 (d) 5

Ans. 2: (d)

Solution:

Fluid to be given (in mL) $= 7.5kg \times 40\dfrac{mL}{kg} = 300\,mL$

Rate (in mL/minute) $= 20 \times 0.05 = 1\,mL/\text{minute}$

$\therefore$ Time required $= \dfrac{300\,ml}{1\,ml/\text{minute}} = 300\,\text{minute} = 5\,\text{hours}$.

Q3. Shyam spent half of his money and was left with as many as he had rupees before, but with half as many rupees as he had paise before. Which of the following is a possible amount of money he is left with?

(a) 49 rupees and 98 paise
(b) 49 rupees and 99 paise
(c) 99 rupees and 99 paise
(d) 99 rupees and 98 paise

Ans. 3: (b)

Solution: Let original amount be $Rs.\ xy$ and ab paise i.e., $Rs.\ xy.ab$.

As given in question, $\frac{1}{2}(xy.ab)=\frac{ab}{2}.xy$

Solving it we get, $98xy = 99ab$

$\therefore\ xy = 99\ \&\ ab = 98$ [$\because$ No common factor b/ω $99\ \&\ 98$]

So original amount $= Rs\ 99$ and 98 paise

$\therefore$ amount left $=\frac{1}{2}(Rs.\ 99\ \&\ 98\ \text{price}) = Rs. 49$ and 99 paise

Q4. How many integers in the set $\{1,2,3,.......,100\}$ have exactly 3 divisors?

(a) 4 (b) 12 (c) 5 (d) 9

Ans. 4: (a)

Solution: Only four integers namely $4,9,25,49$ will have three divisors.

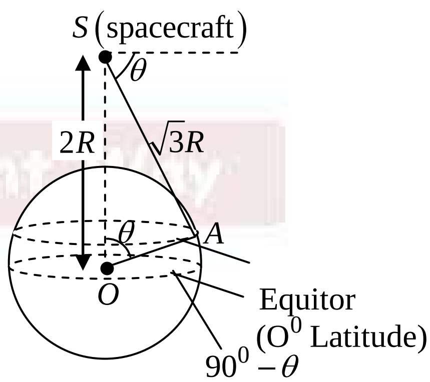
Integers	Divisors
4	1, 2, 4
9	1, 3, 9
25	1, 5, 25
49	1, 7, 49

Q5. A spacecraft flies at a constant height R above a planet of radius R. At the instant the spacecraft is over the north-pole, the lowest latitude visible from the spacecraft is:

(a) 0° (Equator) (b) $30^\circ N$ (c) $45^\circ N$ (d) $60^\circ N$

Ans. 5: (b)

Solution: From the figure, it is obvious that in ΔOAS



$$\tan\theta = \frac{SA}{OA} = \frac{\sqrt{3}R}{R} = \sqrt{3}$$

$\therefore\ \theta = 60^\circ$

$\therefore$ Lowest latitude visible from spacecraft

$= 90^\circ - \theta = 90^\circ - 60^\circ = 30^\circ\ N$

Q6. A and B start from the same point in opposite directions along a circular track simultaneously. Speed of B is $2/3^{rd}$ that of A. How many times will A and B cross each other before meeting at the starting point?

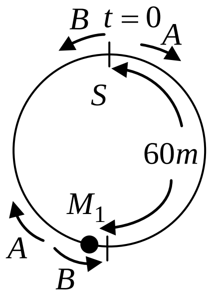
(a) 2 (b) 3 (c) 5 (d) 4

Ans. 6: (d)

Solution: Let the speed of $A = 30\,m/s$ and then speed of $B = 20\,m/s$.

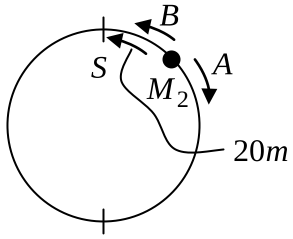
Let again, length of track $= 100\,m$

First Meet



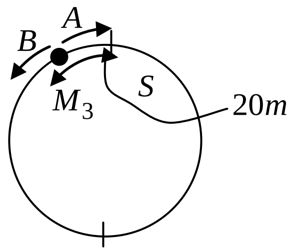
40 meters from study point (s) anti-clock wise

Second Meet



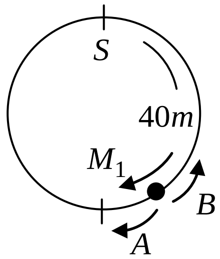
20 meters from study point (s) clock wise

Third Meet



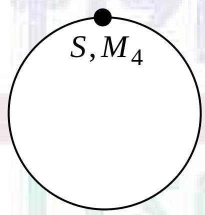
20 meters from study point (s) clock wise

Fourth Meet



40 meters from study point (s) anti-clock wise

Fifth Meet



40 meters from study point (s) anti-clock wise

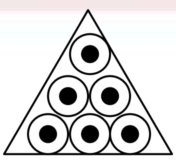
$\therefore$ Total number of cross before meeting at starting point $= 4$.

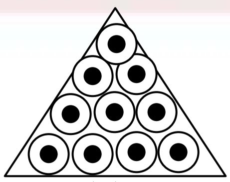
Q7. Identical balls are tightly arranged in the shape of an equilateral triangle with each side containing n balls. How many balls are there in the arrangement?

(a) $n^2/2$ (b) $n(n+1)/2$ (c) $n(n-1)/2$ (d) $(n+1)^2/2$

Ans. 7: (b)

Solution: for $n = 3$ for $n = 4$





No. of balls $f(3) = 6$ $f(4) = 10$

Where $f(x) =$ no. of balls in the arrangements

Let $f(x) = \dfrac{x(x+1)}{2}$ be true for x balls

Then, $f(x+1)=\dfrac{(x+1)(x+2)}{2}=\dfrac{x(x+1)+x+x+2}{2}=\dfrac{x(x+1)}{2}+(x+1)=f(x)+(x+1)$

$\therefore$ If $f(3)=6, f(4)=f(3)+4=6+4=1$

$f(5)=f(4)+5=10+5=15$

Hence it is true for all x.

Q8. An experiment consists of tossing a coin 20 times. Such an experiment is performed 50 times. The number of heads and the number of tails in each experiment are noted. What is the correlation coefficient between the two?

(a) -1 (b) $-20/50$ (c) $20/50$ (d) 1

Ans. 8: (a)

Solution: Let X = no. of heads, Y = no. of tails.

Given $X+Y=x$ (here $x=20\times 50$)

$E[X+Y]=E(x)=x \quad \therefore E(X)+E(Y)=x$

or, $X-E(X)+Y-E(Y)=0$

$\therefore X-E(X)=-(Y-E(Y))$

$Cov(X,Y)=E\left[(X-E(X))(Y-E(Y))\right]=-E\left[(X-E(X))^2\right]=-\mathrm{var}(X)$

Also, $\mathrm{var}(X)=\mathrm{var}(Y)$

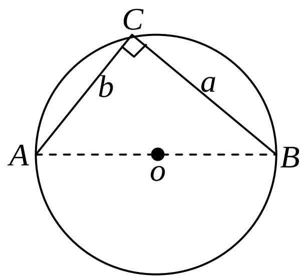
ρ_{XY} (Correlation coefficient) $=\dfrac{\mathrm{Cov}(X,Y)}{\sigma_X\sigma_Y}=\dfrac{-\mathrm{var}(X)}{\sqrt{\mathrm{var}(X)}\sqrt{\mathrm{var}(Y)}}=\dfrac{-\mathrm{var}(X)}{\mathrm{var}(X)}=-1$.

Q9. The maximum area of a right-angled triangle inscribed in a circle of radius r is

(a) $2r^2$ (b) $r^2/2$ (c) $\sqrt{2}r^2$ (d) r^2

Ans. 9: (d)

Solution: Any right –angled triangle ascribed a circle will have hypotenuse coinciding with diameter. Let the side length



$BC=a$ and $AC=b$ and $AB=2r$ (diameter)

$\therefore a^2+b^2=(2r)^2 \Rightarrow a^2+b^2=4r^2$

$(a-b)^2+2ab=4r^2$

$2ab=4r^2-(a-b)^2$ (i)

Also, $\mathrm{area}(\Delta ABC)=ab/2$

$\therefore \max\{ar(\Delta ABC)\} = \max(ab)$ (ii)

So, from Eq. (i)

$2ab = 4r^2 - (a-b)^2$ (iii)

$\max(ab)$ is possible when $(a-b)^2$ is minimum, as $4r^2$ is constant.

$\therefore \min\{(a-b)^2\} = 0$ when $a = b$.

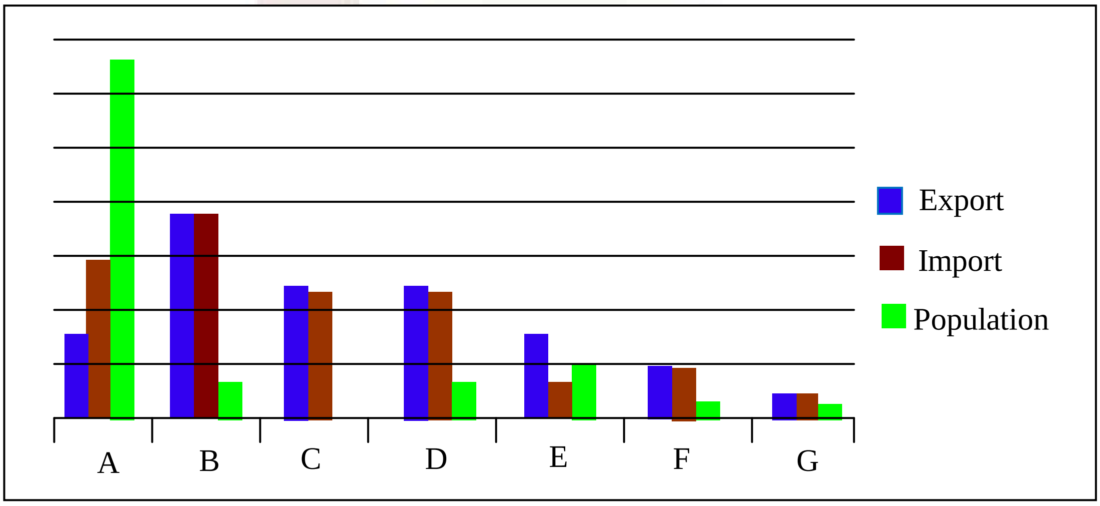
Hence, from (iii),

$$2a^2 = 4r^2$$

or, $a = \sqrt{2}r = b$

$\therefore$ Maximum area $\frac{1}{2}ab = \frac{1}{2}(\sqrt{2}r)(\sqrt{2}r) = r^2$.

Q10. Trade figures populations in appropriate units in a certain year are given for 7 countries.



If countries are ranked according to the difference in their per capita exports over import, the best and worst ranking countries are respectively.

(a) C and A (b) A and E (c) C and B (d) A and F

Ans. 10: (a)

Solution:

Rank determinant formula $= \frac{\text{Export-Import}}{\text{Population}}$

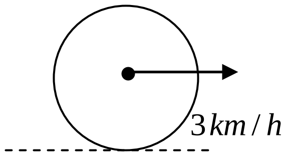
It is obvious that in case of (c), population is mineral compared to difference between export and import. Hence highest rank, in case of A: Export $<$ Import with highest population, so it will have lowest rank.

Q11. A cylindrical road roller having a diameter of $1.5m$ moves at a speed of $3km/h$ while levelling a road. How much length of the road will be leveled in 45 minutes?

(a) $2.25\,km$ (b) $0.375\pi km$ (c) $0.75\pi km$ (d) $1.5\,kn$

Ans. 11: (a)

Solution: . Distance covered by roller in 45 minutes at a speed of



$$3\,km/h = (3km/h)\left(\frac{45}{60}h\right) = 3\times\frac{45}{60}km$$

$$= 2.25\,km$$

Q12. Which of these groups of numbers has the smallest mean?

Group A: $1,2,3,4,5,6,7,8,9$

Group B: $1,2,3,4,6,6,7,8,9$

Group C: $1,2,2,4,5,6,7,8,9$

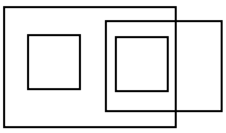
Group D: $1,3,3,4,5,6,7,9,9$

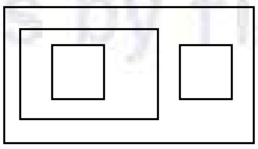
(a) A (b) B (c) C (d) D

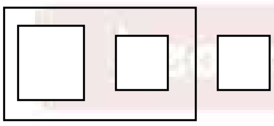
Ans. 12: (c)

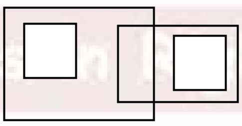
Solution: Smallest mean is of group $C = \dfrac{1+2+2+4+5+6+7+8+9}{9} = \dfrac{44}{9} \approx 4.88$

Q13. An appropriate diagram to represent the relations between the categories KEYBOARD, HARDWARE, OPERATING SYSTEM and CPU is

(a) 

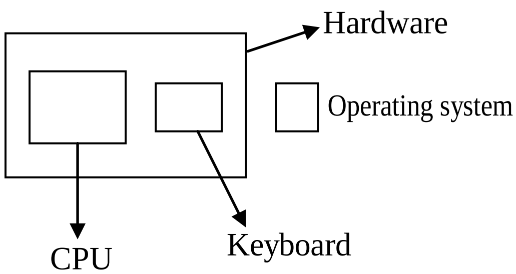
(b) 

(c) 

(d) 

Ans. 13: (c)

Solution:



Q14. If we replace the mathematical operations in the expression $(11+4-2)\div 24\times 6$ as given in the table:

Operation	$+$	$-$	$\times$	$\div$
Replaced by	$-$	$\times$	$\div$	$+$

Then is new value is

(a) $23/6$ (b) 1 (c) 18 (d) 7

Ans. 14: (d)

Solution: $(11+4-2)\div 24\times 6$

$$\downarrow \quad \downarrow \quad \downarrow \quad \downarrow$$

$$(11-4\times 2)+24\div 6 \quad \Rightarrow (11-8)+4=3+4=7$$

Q15. In a tournament with 8 teams, a win fetches 3 points and a draw, 1. After all teams have played three matches each, total number of points earned by all teams put together must lie between

(a) 24 and 36 (b) 24 and 32 (c) 12 and 24 (d) 32 and 48

Ans. 15: (a)

Solution: Total number of matches played $=8\times 3=24$

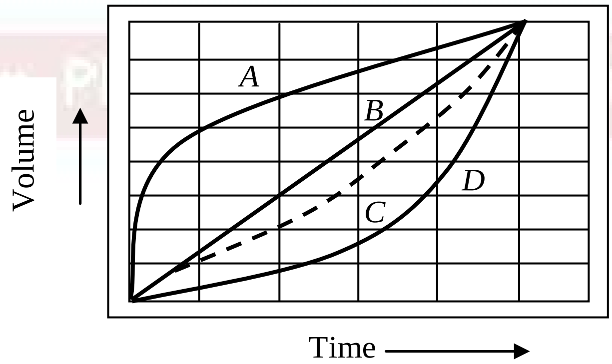
If all matches are either win or lost then, Total number of win $=12$

Total number of loose $=12$ $\quad\therefore$ In this case total points $==12\times 3=36$ points.

Next, if all matches are draw, then Points earned by all terms $=24\times 1=24$ points

$\therefore$ Total number of points will lie between 24 and 36.

Q16. An inverted cone is filled with water at a constant rate. The volume of water inside the cone as a function of times is represented the curve



(a) A (b) B (c) C (d) D

Ans. 16: (b)

Solution: Volume of water is being poured with constant rate hence the curve value v time will be linear one. i.e., B

Q17. At least two among three persons A, B and C are truthful. If A calls B a liar and if B calls C a liar, then which of the following is FALSE?

(a) A is truthful (b) B is truthful

(c) C is truthful (d) At least one is a liar

Ans. 17: (b)

Solution: Suppose B is truthful. Then from the statements of question A is liar.

Also, B calls C is lia $\Rightarrow C$ is liar.

But at least two of A, B, C are truthful.

Q18. A shopkeeper has a faulty pan balance with a zero offset. When an object is placed in the left plan it is balanced by a standard $100g$ weight. When it is placed in the right pan it is balanced by a standard $80g$ weight. What is the actual weight of the object?

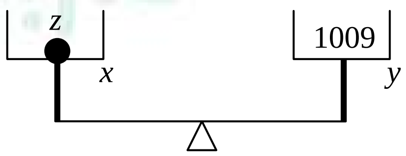
(a) $90g$ (b) $88.88g$ (c) $95g$ (d) $85g$

Ans. 18: (a)

Solution: Let the weight of left and right pans be x & y gm and the object weight $= z\,g$.

From fig (a),

$$z + x = 100 + y \qquad \text{(i)}$$



from fig (b),

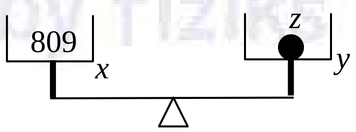
$$80 + x = z + y \qquad \text{(ii)}$$

(i) –(ii) gives.

$z - 80 = 100 - z$

or $2z = 180$

$\therefore z = 90\,gm$



Q19. A cousin is a non-sibling with a common ancestor. If there is exactly one pair of siblings in a group of 5 persons then the maximum possible number of pairs of cousins in the group is

(a) 3 (b) 6 (c) 9 (d) 10

Ans. 19: (c)

Solution: Total number of relations (pairs) $= {}^5C_2 = 10$

Of these one is of siblings.

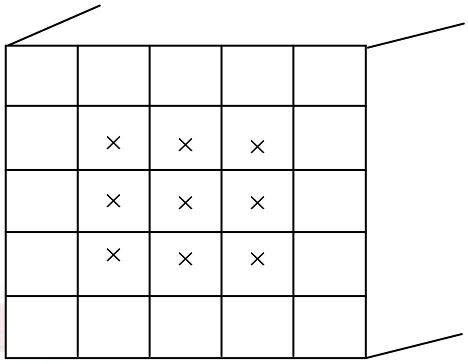
$\therefore$ Maximum possible number of pairs of cousins in the group $= 10 - 1 = 9$.

Q20. Consider a solid cube of side 5 units. After painting, it is cut into cubes of 1 unit. Find the probability that a randomly chosen unit cube has only one side painted.

(a) $56/125$ (b) $36/125$ (c) $44/125$ (d) $54/125$

Ans. 20: (d)

Solution: Total number of cubes of unit size $=125$



On one face, we set 9 cubes of one face painted.

So, total cubes of unit size with one face painted

$$=9\times 6\ =54$$

$\therefore$ Required probability $\dfrac{54}{125}$

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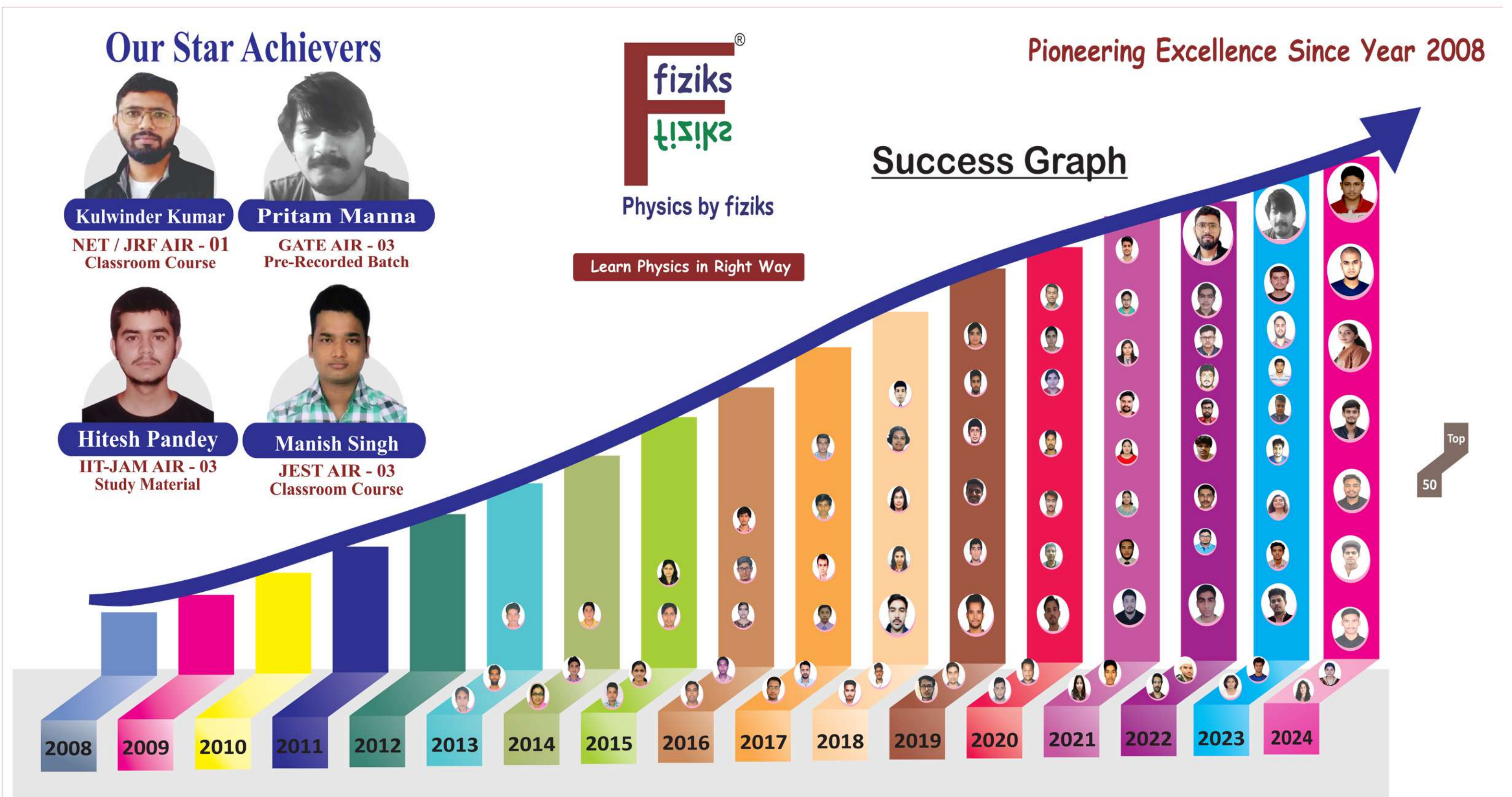

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