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Learn Physics in Right Way

CSIR NET-JRF Physical Sciences Paper Sep-2022 Solution

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Part-B

Q22. Two $n\times n$ invertible real matrices A and B satisfy the relation

$$(AB)^T = -\left(A^{-1}B\right)^{-1}$$

If B is orthogonal then A must be

(a) Lower triangular (b) Orthogonal

(c) Symmetric (d) Antisymmetric

Ans.:(d)

Solution:

Given that B is an orthogonal matrix then $BB^T = I$.

$$\because (AB)^T = -\left(A^{-1}B\right)^{-1} = -B^{-1}\left(A^{-1}\right)^{-1} = -B^{-1}A \Rightarrow B^T A^T = -B^{-1}A \Rightarrow BB^T A^T = -BB^{-1}A$$

$$\Rightarrow IA^T = -IA \quad \Rightarrow A^T = -A$$

Q23. The value of the integral $\int_0^\infty dx\, e^{-x^{2m}}$, where m is a positive integer, is

(a) $\Gamma\left(\frac{m+1}{2m}\right)$ (b) $\Gamma\left(\frac{m-1}{2m}\right)$

(c) $\Gamma\left(\frac{2m+1}{2m}\right)$ (d) $\Gamma\left(\frac{2m-1}{2m}\right)$

Ans.:(c)

Solution: $\because \int_0^\infty x^m e^{-\alpha x^n} dx = \frac{1}{n}\frac{\left\lceil \frac{m+1}{n} \right.}{\alpha^{m+1/n}}$

Thus $\int_0^\infty dx\, e^{-x^{2m}} = \frac{1}{2m}\frac{\left\lceil \frac{0+1}{2m}\right.}{(1)^{0+1/2m}} = \frac{1}{2m}\left\lceil \frac{1}{2m}\right. = \left\lceil \frac{1}{2m}+1\right. = \left\lceil \frac{1+2m}{2m}\right.$ where $n = 2m,\ m = 0,\ \alpha = 1$

Q26. If $z = i^{i^{i^{\cdot^{\cdot}}}}$ (note that the exponent continues indefinitely), then a possible value of $\frac{1}{z}\ln z$ is

(a) $2i\ln i$ (b) $\ln i$

(c) $i\ln i$ (d) $2\ln i$

Ans.:(b)

Solution: $\because z = i^{i^{i^{\cdot^{\cdot}}}} \Rightarrow z = i^z$

Take log of both side; $\ln z = z\ln i \Rightarrow \frac{1}{z}\ln z = \ln i$

Q40. At $z=0$, the function $\dfrac{1}{z-\sin z}$ of a complex variable z has

(a) no singularity (b) a simple pole

(c) a pole of order 2 (d) a pole of order 3

Ans.:(d)

Solution: $f(z)=\dfrac{1}{z-\sin z}=\dfrac{1}{z-\left(z-\dfrac{z^3}{3!}+\dfrac{z^5}{5!}-\dots\right)}=\dfrac{1}{\dfrac{z^3}{3!}-\dfrac{z^5}{5!}+\dots}=\dfrac{1}{\dfrac{z^3}{3!}\left(1-\dfrac{3!}{5!}z^2+\dots\right)}$

$$\Rightarrow f(z)=\frac{3!}{z^3}\left(1-\frac{3!}{5!}z^2+\dots\right)^{-1}=\frac{3!}{z^3}\left(1+\frac{3!}{5!}z^2-\dots\right)=3!z^{-3}++\frac{(3!)^{-1}}{5!}z^{-1}-\dots$$

$f(z)$ has a pole of order 3.

Q41. The infinite series $\sum_{n=0}^{\infty}\left(n^2+3n+2\right)x^n$ evaluated at $x=\dfrac{1}{2}$, is

(a) 16 (b) 32 (c) 8 (d) 24

Ans. :(a)

Solution: $\sum_{n=0}^{\infty}\left(n^2+3n+2\right)x^n=2+6x+12x^2+\dots=2\left(1+3x+6x^2+\dots\right)=2(1-x)^{-3}$

At $x=\dfrac{1}{2}$: $2(1-x)^{-3}=2\left(1-\dfrac{1}{2}\right)^{-3}=2\times2^3=16$

Q42. A walker takes steps, each of length L, randomly in the directions along east, west, north and south. After four steps its distance from the starting point is d. The probability that $d\le 3L$ is

(a) 63/64 (b) 59/64 (c) 57/64 (d) 55/64

Ans.: (d)

Solution:

In 4-steps, the positions at which walker could be are:

$(0,0)$,

$(L,L),(-L,L),(L,-L),(-L,-L)$

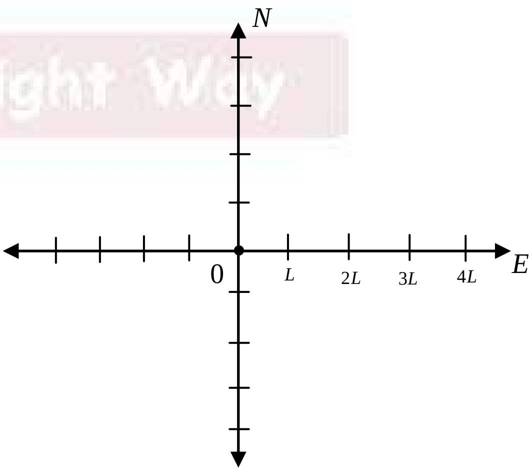
$(2L,0),(0,2L),(-2L,0),(0,-2L)$

$(2L,2L),(-2L,2L),(-2L,-2L),(2L,-2L)$

$(3L,L),(3L,-L),(-3L,L),(-3L,-L)$

$(L,3L),(-L,3L),(L,-3L),(-L,-3L)$

$(4L,0),(-4L,0),(0,4L),(0,-4L)$



Out of these listings, two rows violates the distance constraint of $d \le 3L$:

$(3L, L),(3L,-L),(-3L, L),(-3L,-L)$ $(L, 3L),(-L, 3L),(L,-3L),(-L,-3L)$

$(4L, 0),(-4L, 0),(0, 4L),(0,-4L)$

Probability of walker being at position $(4L,0)=\dfrac{1}{256}$

Using Symmetry:

Probability of walker being at $(4L,0)$ or $(-4L,0)$ or $(0,4L)$ or $(0,-4L)=\dfrac{4}{256}$

Next, Probability of walker being at position $(3L,L)=\dfrac{{}^4C_3\cdot{}^1C_1}{256}$

Using symmetry:

Probability of walker being at $(3L,L)$ or $(3L,-L)$ or $(-3L,L)$ or $(-3L,-L)$ or $(L,3L)$ $(-L,3L)$ or $(L,-3L)$ or $(-L,-3L)$ is $=8\cdot\dfrac{{}^4C_3\cdot{}^1C_1}{256}=\dfrac{32}{256}$

$$\therefore\ P(d\le 3L)=1-\left(\frac{4}{256}+\frac{32}{256}\right)=1-\frac{36}{256}=\frac{220}{256}=\frac{55}{64}$$

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Part-C

Q54. The matrix corresponding to the differential operator $\left(1+\frac{d}{dx}\right)$ in the space of polynomials of degree at most two, in the basis spanned by $f_1 = 1$, $f_2 = x$ and $f_3 = x^2$, is

(a) $\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$ (b) $\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 2 & 1 \end{pmatrix}$ (c) $\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{pmatrix}$ (d) $\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 2 \end{pmatrix}$

Ans.: (a)

Solution:

$$D = \left(1+\frac{d}{dx}\right) \; ; \; f_1 = 1, \; f_2 = x \text{ and } f_3 = x^2$$

Now $X = Df = \left(1+\frac{d}{dx}\right) f \Rightarrow C_0 + C_1 x + C_2 x^2 = \left(1+\frac{d}{dx}\right) f$

Let $f = f_1 = 1 \Rightarrow C_0 + C_1 x + C_2 x^2 = \left(1+\frac{d}{dx}\right) 1 = 1 \Rightarrow C_0 = 1, C_1 = 0, C_2 = 0$

$$f = f_2 = x \Rightarrow C_0 + C_1 x + C_2 x^2 = \left(1+\frac{d}{dx}\right) x = x+1 \Rightarrow C_0 = 1, C_1 = 1, C_2 = 0$$

$$f = f_2 = x^2 \Rightarrow C_0 + C_1 x + C_2 x^2 = \left(1+\frac{d}{dx}\right) x^2 = x^2 + 2x \Rightarrow C_0 = 0, C_1 = 2, C_2 = 1$$

$$\Rightarrow X_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad X_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad X_3 = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}. \text{ Thus } \Rightarrow X = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

Thus $X = Df = \left(1+\frac{d}{dx}\right) f$

Q59. The value of the integral $\int_{-\infty}^{\infty} \frac{\cos \alpha x}{x^2+1} dx$, for $\alpha > 0$, is

(a) πe^{α} (b) $\pi e^{-\alpha}$

(c) $\pi e^{-\alpha/2}$ (d) $\pi e^{\alpha/2}$

Ans. :(b)

Solution:

$$\because \int_0^{\infty} \frac{\cos mx}{x^2+a^2} dx = \frac{\pi}{2a} e^{-ma}, \; m \geq 0$$

$$\int_{-\infty}^{\infty} \frac{\cos \alpha x}{x^2+1} dx = 2\int_0^{\infty} \frac{\cos \alpha x}{x^2+1} dx = 2 \times \frac{\pi}{2} e^{-\alpha} = \pi e^{-\alpha}$$

Q68. The Laplace transform $L[f](y)$ of the function $f(x)=\begin{cases} 1 & \text{for } 2n \le x \le 2n+1 \\ 0 & \text{for } 2n+1 \le x \le 2n+2 \end{cases}$,

$n=0,1,2,....$ is

(a) $\dfrac{e^{-y}\left(e^{-y}+1\right)}{y\left(e^{-2y}+1\right)}$ (b) $\dfrac{e^{y}-e^{-y}}{y}$ (c) $\dfrac{e^{y}+e^{-y}}{y}$ (d) $\dfrac{e^{y}\left(e^{y}-1\right)}{y\left(e^{2y}-1\right)}$

Ans.:(d)

Solution: Laplace transform of periodic function is given by $L\left[f(x)\right]=\dfrac{1}{1-e^{-yT}}\int_0^T f(x)e^{-yx}d$

For $n=0$, $T=2$;

$$L[f](y)=\frac{1}{1-e^{-2y}}\int_0^2 f(x)e^{-yx}dx=\frac{1}{1-e^{-2y}}\int_0^1 1.e^{-yx}dx=\frac{1}{1-e^{-2y}}\left[\frac{e^{-yx}}{-y}\right]_0^1$$

$$\Rightarrow L[f](y)=\frac{-1}{y\left(1-e^{-2y}\right)}\left[e^{-y}-1\right]=\frac{\left(1-e^{-y}\right)}{ye^{-2y}\left(e^{2y}-1\right)}=\frac{e^{2y}\left(1-e^{-y}\right)}{y\left(e^{2y}-1\right)}=\frac{e^{2y}e^{-y}\left(e^{y}-1\right)}{y\left(e^{2y}-1\right)}$$

$$\Rightarrow L[f](y)=\frac{e^{y}\left(e^{y}-1\right)}{y\left(e^{2y}-1\right)}$$

Q71. A bucket contains 6 red and 4 blue balls. A ball is taken out of the bucket at random and two balls of the same colour are put back. This step is repeated once more. The probability that the numbers of red and blue balls are equal at the end, is

(a) 4/11 (b) 2/11 (c) 1/4 (d) 3/4

Ans.: (b)

Solution: No. of Red balls = 6; No. of Blue balls = 4

	R(6)	B(4)
1st step:	6	4 – 1 + 2 = 5
2nd step:	6	5 – 1 + 2 = 6

Required Prob. = Probability of performing 1st Step × Prob. of Performing 2nd Step

Probability of Performing 1st Step = Probability of Picking a blue ball $=\dfrac{4}{10}$

Probability of Performing 2nd Step = Probability of Picking again a blue ball $=\dfrac{5}{11}$

$\therefore$ Required Probability $=\dfrac{4}{10}\times\dfrac{5}{11}=\dfrac{2}{11}$

Part-B

Q28. A particle of rest mass m is moving with a velocity $v\hat{k}$, with respect to an inertial frame S. The energy of the particle as measured by an observer S', who is moving with a uniform velocity $u\hat{i}$ with respect to S (in terms of $\gamma_u = 1/\sqrt{1-u^2/c^2}$ and $\gamma_v = 1/\sqrt{1-v^2/c^2}$ is

(a) $\gamma_u \gamma_v m\left(c^2 - uv\right)$

(b) $\gamma_u \gamma_v mc^2$

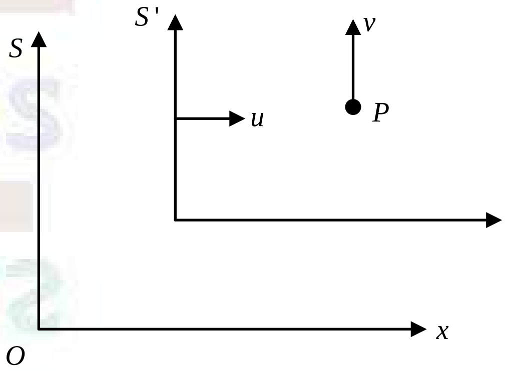
(c) $\frac{1}{2}\left(\gamma_u + \gamma_v\right)mc^2$

(d) $\frac{1}{2}\left(\gamma_u + \gamma_v\right)m\left(c^2 - uv\right)$

Ans.:(b)

Solution :

$\vec{v}_P = 0\hat{i} + 0\hat{j} + v\hat{k}$



In S-frame:

$$E = \frac{mc^2}{\sqrt{1-\frac{v^2}{c^2}}} = \gamma_v mc^2 ; \ \vec{p} = 0\hat{i} + 0\hat{j} + \frac{mv}{\sqrt{1-\frac{v^2}{c^2}}}\hat{k} = \gamma_v mv\hat{k}$$

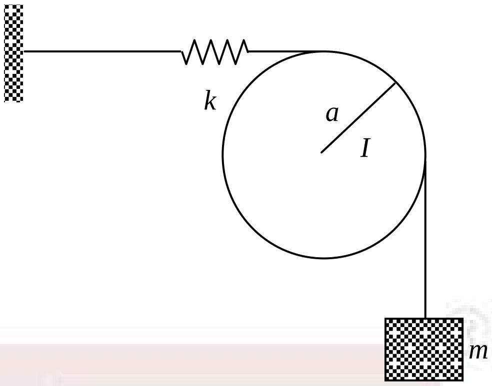
In S'-frame:

$$p'_x = \frac{p_x - uE/c^2}{\sqrt{1-\frac{u^2}{c^2}}} = \frac{0 - uE/c^2}{\sqrt{1-\frac{u^2}{c^2}}} = -\gamma_u \frac{uE}{c^2} ; \ \ p'_y = p_y = 0$$

$$p'_z = p_z = \frac{mv}{\sqrt{1-\frac{v^2}{c^2}}} = \gamma_v mv$$

$$E' = \frac{E - up_x}{\sqrt{1-\frac{u^2}{c^2}}} = \frac{E-0}{\sqrt{1-\frac{u^2}{c^2}}} = \gamma_u E \ \ \Rightarrow E' = \gamma_u \gamma_v mc^2$$

Q29. A wire, connected to a massless spring of spring constant k and a block of mass m, goes around a disc of radius a and moment of inertia I, as shown in the figure.



Assume that the spring remains horizontal, the pulley rotates freely and there is no slippage between the wire and the pulley. The angular frequency of small oscillations of the disc is

(a) $\sqrt{\dfrac{2ka^2}{ma^2+I}}$ (b) $\sqrt{\dfrac{ka^2}{ma^2+I}}$

(c) $\sqrt{\dfrac{ka^2}{ma^2+2I}}$ (d) $\sqrt{\dfrac{ka^2}{2ma^2+I}}$

Ans.: (b)

Solution: In equilibrium $mg = T = kx_0$

If mass is displaced in downward direction by a small distance x, with respect to equilibrium position.

$$T = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}I\omega^2 = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}I\frac{\dot{x}^2}{a^2} \Rightarrow T = \frac{1}{2}\left(m + \frac{I}{a^2}\right)\dot{x}^2 \text{ and } V = \frac{1}{2}k\left(x + x_0\right)^2 - mgx$$

Here gravitational potential energy is calculated with respect to equilibrium position

$$L = T - V = \frac{1}{2}\left(m + \frac{I}{a^2}\right)\dot{x}^2 - \frac{1}{2}k\left(x + x_0\right)^2 + mgx$$

$$\frac{d}{d}\left(\frac{\partial L}{\partial \dot{x}}\right) = \left(m + \frac{I}{a^2}\right)\ddot{x}; \quad \frac{\partial L}{\partial x} = -k\left(x + \cancel{x_0}\right) + \cancel{mg} = -kx$$

Lagrange's Equation of motion $\left(m + \dfrac{I}{a^2}\right)\ddot{x} + kx = 0 \Rightarrow \ddot{x} + \dfrac{k}{\left(m + \dfrac{I}{a^2}\right)}x = 0$

$$\Rightarrow \omega = \sqrt{\frac{k}{m + \dfrac{I}{a^2}}} = \sqrt{\frac{ka^2}{ma^2 + I}}$$

Q36. The Lagrangian of a system described by three generalized coordinates q_1, q_2 and q_3 is $L = \frac{1}{2}m\left(\dot{q}_1^2 + \dot{q}_2^2\right) + M\dot{q}_1\dot{q}_2 + k\dot{q}_1 q_3$, where m, M and k are positive constants. Then, as a function of time

(a) two coordinates remain constant and one evolves linearly

(b) one coordinate remains constant, one evolves linearly and the third evolves as a quadratic function

(c) one coordinate evolves linearly and two evolve quadratically

(d) all three evolve linearly

Ans.: (a)

Solution: q_2 and q_3 are cyclic coordinates so,

$$p_1 = \frac{\partial L}{\partial \dot{q}_1} = m\dot{q}_1 + m\dot{q}_2 + kq_3 = \text{constant} \quad \text{....(1)}$$

$$p_2 = \frac{\partial L}{\partial \dot{q}_2} = m\dot{q}_2 + m\dot{q}_1 = \text{constant} \quad \text{....(2)}$$

Lagrange's equation of motion for q_3; $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_3}\right) - \frac{\partial L}{\partial q_3} = 0$

$$0 - k\dot{q}_1 = 0 \rightarrow \dot{q}_1 = 0 \rightarrow \boxed{q_1 = \text{constant}}$$

From Equation (2); $m\dot{q}_2 + 0 = \text{constant} \rightarrow \boxed{q_2 = ct + d}$

From Equation (1); $0 + \text{constant} + kq_3 = \text{constant} \Rightarrow \boxed{q_3 = \text{constant}}$

Q44. The periods of oscillation of a simple pendulum at the sea level and at the top of a mountain of height 6 km are T_1 and T_2, respectively. If the radius of earth is approximately 6000 km, then $\frac{(T_2 - T_1)}{T_1}$ is closest to

(a) -10^{-4} (b) -10^{-3}

(c) 10^{-4} (d) 10^{-3}

Ans.: (d)

Solution: $T_1 = 2\pi\sqrt{\frac{l}{g}}$, $g' = \frac{g}{\left(1+\frac{h}{R}\right)^2}$

$$T_2 = 2\pi\sqrt{\frac{l}{g'}} = \left(1+\frac{h}{R}\right)2\pi\sqrt{\frac{l}{g}} \Rightarrow T_2 = \left(1+\frac{h}{R}\right)T_1 \Rightarrow \frac{T_2 - T_1}{T_1} = \frac{h}{R} = \frac{6}{6000} = 10^{-3}$$

Part-C

Q48. Earth may be assumed to be an axially symmetric freely rotating rigid body. The ratio of the principal moments of inertia about the axis of symmetry and an axis perpendicular to it is 33:32. If T_0 is the time taken by earth to make one rotation around its axis of symmetry, then the time period of precession is closest to

(a) $33T_0$ (b) $33T_0/2$

(c) $32T_0$ (d) $16T_0$

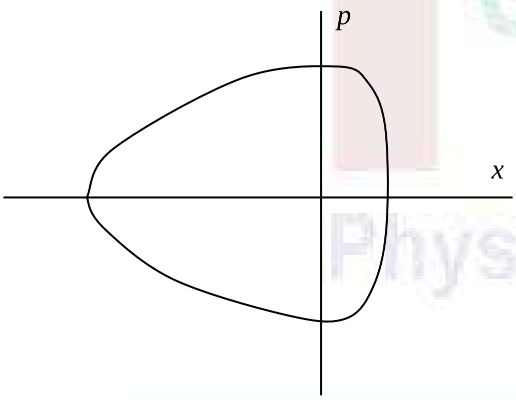
Ans.: (c)

Solution:

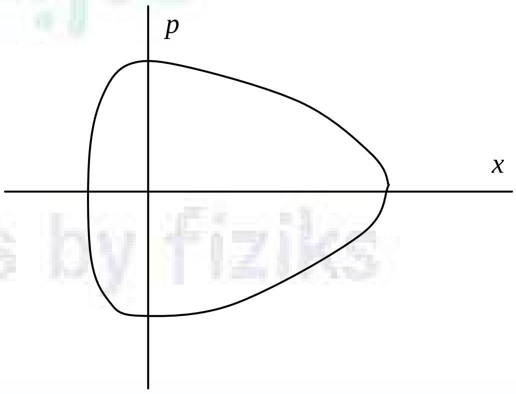
$$\Omega = \frac{I_3 - I_1}{I_1}\omega_3 \Rightarrow \frac{2\pi}{T} = \frac{\frac{33}{32}-1}{1}\frac{2\pi}{T_0} \Rightarrow T = 32T_0 \qquad \because \frac{I_3}{I_1} = \frac{33}{32}$$

Q60. The Lagrangian of a particle in one dimension is $L = \frac{m}{2}\dot{x}^2 - ax^2 - V_0 e^{-10x}$ where a and V_0 are positive constants. The best qualitative representation of a trajectory in the phase space is

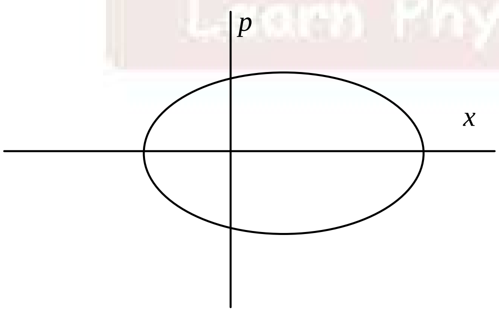
(a)



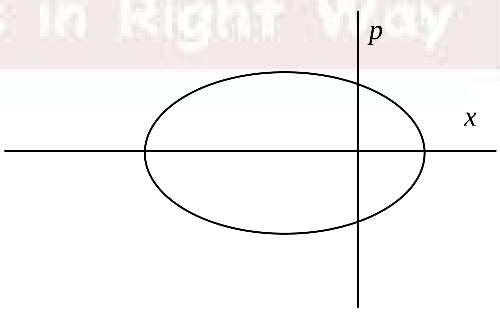
(b)



(c)



(d)

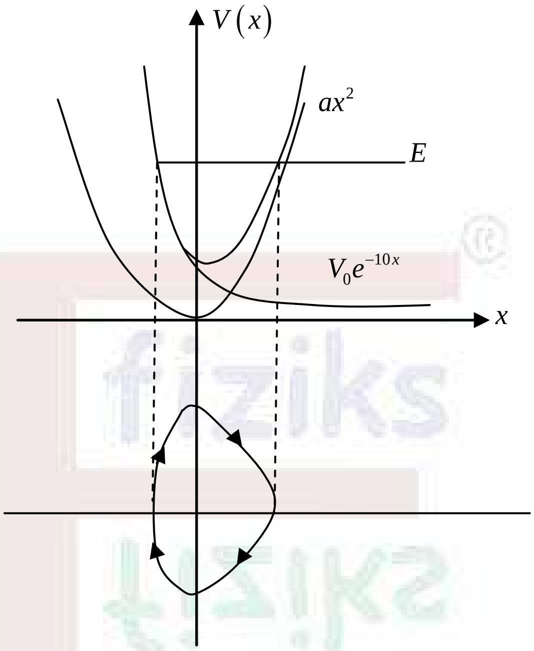


Ans.: (b)

Solution:

$V(x) = ax^2 + V_0 e^{-10x}$

Plot ax^2 and $V_0 e^{-10x}$ separately on the same axis system and combine them together to plot the phase curve.



Q67. The Lagrangian of system of two particles is $L = \frac{1}{2}\dot{x}_1^2 + 2\dot{x}_2^2 - \frac{1}{2}\left(x_1^2 + x_2^2 + x_1 x_2\right)$. The normal frequencies are best approximated by

(a) 1.2 and 0.7
(b) 1.5 and 0.5
(c) 1.7 and 0.5
(d) 1.0 and 0.4

Ans.: (d)

Solution:

$\because L = \frac{1}{2}\dot{x}_1^2 + 2\dot{x}_2^2 - \frac{1}{2}\left(x_1^2 + x_2^2 + x_1 x_2\right)$

$\Rightarrow T = \frac{1}{2}\dot{x}_1^2 + 2\dot{x}_2^2 = \frac{1}{2}\dot{x}_1^2 + \frac{1}{2}4\dot{x}_2^2 \Rightarrow T = \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}$

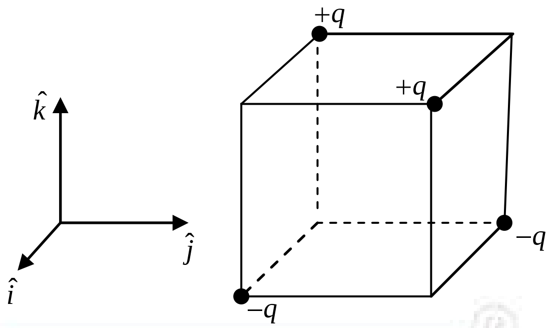
$\Rightarrow V = \frac{1}{2}\left(x_1^2 + x_2^2 + x_1 x_2\right) = \frac{1}{2}\left(x_1^2 + x_2^2 + \frac{1}{2}x_1 x_2 + \frac{1}{2}x_2 x_1\right) \Rightarrow T = \begin{pmatrix} 1 & 1/2 \\ 1/2 & 1 \end{pmatrix}$

Secular Equation: $\left|V - \omega^2 T\right| = 0 \Rightarrow \begin{vmatrix} 1-\omega^2 & 1/2 \\ 1/2 & 1-4\omega^2 \end{vmatrix} = 0 \Rightarrow \left(1-\omega^2\right)\left(1-4\omega^2\right) - \frac{1}{4} = 0$

$\Rightarrow 1 - 4\omega^2 - \omega^2 + 4\omega^4 - \frac{1}{4} = 0 \Rightarrow 4\omega^4 - 5\omega^2 + \frac{3}{4} = 0 \Rightarrow 16\omega^4 - 20\omega^2 + 3 = 0 \Rightarrow \omega = 1.03,\ 0.41$

Part-B

Q25. Two positive and two negative charges of magnitude q are placed on the alternate vertices of a cube of side a (as shown in the figure).



The electric dipole moment of this charge configuration is

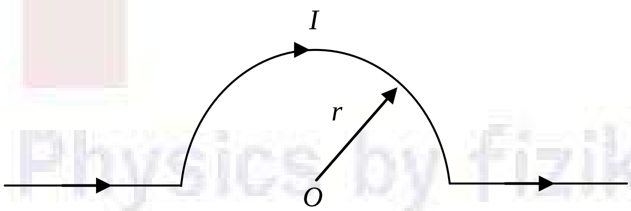
(a) $-2qa\hat{k}$ (b) $2qa\hat{k}$ (c) $2qa\left(\hat{i}+\hat{j}\right)$ (d) $2qa\left(\hat{i}-\hat{j}\right)$

Ans.:(b)

Solution: The electric dipole moment of this charge configuration is

$$\vec{p}=-q\left(a\hat{i}\right)-q\left(a\hat{j}\right)+q\left(a\hat{k}\right)+q\left(a\hat{i}+a\hat{j}+a\hat{k}\right)\Rightarrow\vec{p}=2qa\hat{k}$$

Q30. A part of an infinitely long wire, carrying a current I, is bent in a semi-circular arc of radius r (as shown in the figure).



The magnetic field at the centre O of the arc is

(a) $\frac{\mu_0 I}{4r}$ (b) $\frac{\mu_0 I}{4\pi r}$ (c) $\frac{\mu_0 I}{2r}$ (d) $\frac{\mu_0 I}{2\pi r}$

Ans.:(a)

Q32. An electromagnetic wave is incident from vacuum normally on a planer surface of a non-magnetic medium. If the amplitude of the electric field of the incident wave is E_0 and that of the transmitted wave is $2E_0/3$, then neglecting any loss, the refractive index of the medium is

(a) 1.5 (b) 2.0

(c) 2.4 (d) 2.7

Ans.:(b)

Solution: $E_{0T}=\left(\frac{2n_1}{n_1+n_2}\right)E_{0I}\Rightarrow\frac{2}{3}E_0=\left(\frac{2\times1}{1+n_2}\right)E_0\Rightarrow n_2=2.0$

Q35. The electric and magnetic fields in an inertial frame are $\vec{E} = 3a\hat{i} - 4\hat{j}$ and $\vec{B} = \frac{5a}{c}\hat{k}$, where a is a constant. A massive charged particle is released from rest. The necessary and sufficient condition that there is an inertial frame, where the trajectory of the particle is a uniform-pitched helix, is

(a) $1 < a < \sqrt{2}$ (b) $-1 < a < 1$

(c) $a^2 > 1$ (d) $a^2 > 2$

Ans.: (c)

Part-C

Q46. Two small metallic objects are embedded in a weakly conducting medium of conductivity σ and dielectric constant $\in$. A battery connected between them leads to a potential difference V_0. It is subsequently disconnected at time $t = 0$. The potential difference at a later time t is

(a) $V_0 e^{-\frac{t\sigma}{4\in}}$ (b) $V_0 e^{-\frac{t\sigma}{2\in}}$

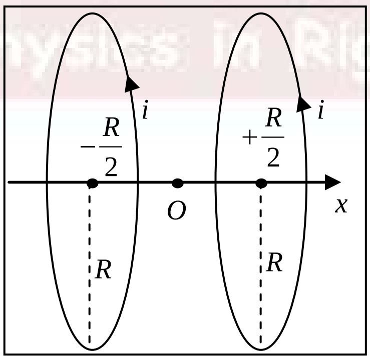
(c) $V_0 e^{-\frac{3t\sigma}{4\in}}$ (d) $V_0 e^{-\frac{t\sigma}{\in}}$

Ans. :(d)

Solution: $I = \oint_S \vec{J}.d\vec{a} = \sigma \oint_S \vec{E}.d\vec{a} = \sigma \frac{q}{\varepsilon_0}$

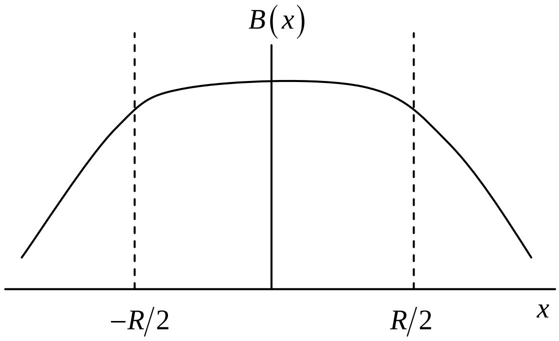
$$\frac{dq}{dt} = -I = -\frac{\sigma}{\varepsilon_0} q \Rightarrow q(t) = q_0 e^{-\frac{\sigma}{\varepsilon_0}t} \Rightarrow V(t) = \frac{q(t)}{C} = V_0 e^{-\frac{\sigma}{\varepsilon_0}t} \qquad \because V_0 = \frac{q_0}{C}$$

Q52. Two parallel conducting rings, both of radius R, are separated by a distance R. The planes of the rings are perpendicular to the line joining their centres, which is taken to be the x-axis.

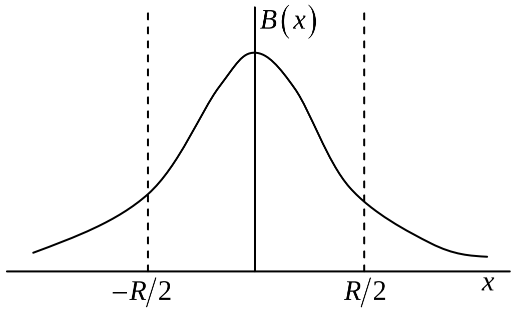


If both the rings carry the same current i along the same direction, the magnitude of the magnetic field along the x-axis is best represented by

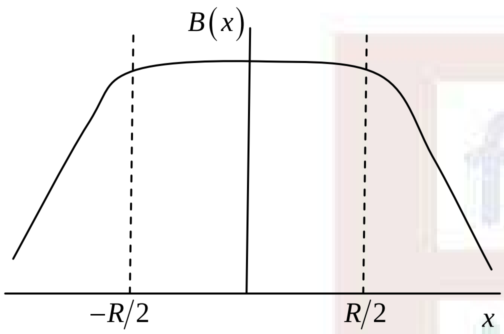
(a)



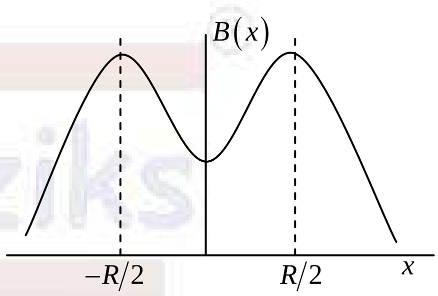
(b)



(c)

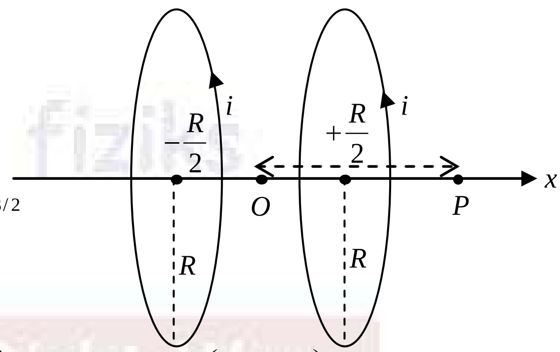


(d)



Ans.: (a)

Solution:



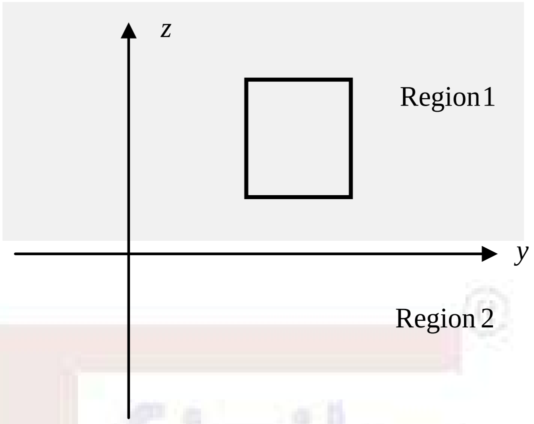
$$B=\frac{\mu_0 I}{2}\frac{R^2}{\left[R^2+\left(x+\frac{R}{2}\right)^2\right]^{3/2}}+\frac{\mu_0 I}{2}\frac{R^2}{\left[R^2+\left(x-\frac{R}{2}\right)^2\right]^{3/2}}$$

At $x=0$; $B=\frac{\mu_0 I}{2}\frac{R^2}{\left[5R^2/2\right]^{3/2}}+\frac{\mu_0 I}{2}\frac{R^2}{\left[5R^2/2\right]^{3/2}}=\frac{\mu_0 I}{R}\left(\frac{2}{5}\right)^{3/2}$

At $x=+\frac{R}{2}$; $B=\frac{\mu_0 I}{2}\frac{R^2}{\left[R^2+\left(\frac{R}{2}+\frac{R}{2}\right)^2\right]^{3/2}}+\frac{\mu_0 I}{2}\frac{R^2}{\left[R^2+\left(\frac{R}{2}-\frac{R}{2}\right)^2\right]^{3/2}}=\frac{\mu_0 I}{2R}\left(\frac{1}{2^{3/2}}+1\right)$

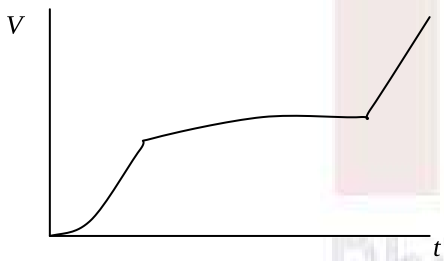
$$\Rightarrow B\left(x=R/2\right)<B\left(x=0\right)$$

Q70. A square conducting loop in the *yz*-plane, falls downward under gravity along the negative *z*-axis. Region 1, defined by $z > 0$ has a uniform magnetic field $\vec{B} = B_0\hat{i}$ while region 2 (defined by $z < 0$) has no magnetic field.

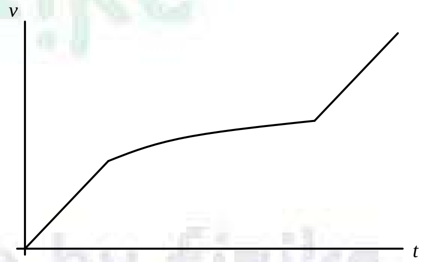


The time dependence of the speed $v(t)$ of the loop, as it starts to fall from well within the region 1 and passes into the region 2, is best represented by

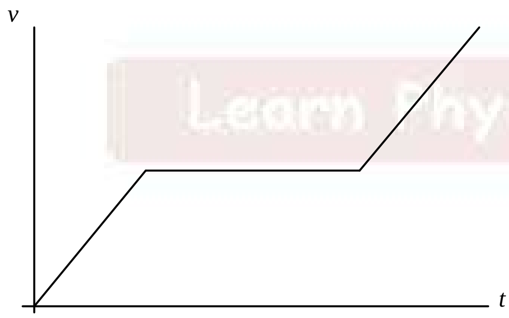
(a)



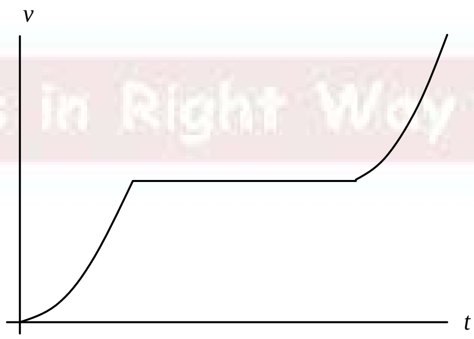
(b)



(c)



(d)



Ans.:(b)

Solution:

Magnetic force experienced by loop is $F = \frac{B^2 l^2 v}{R}$ (upwards)

This force is in the direction of gravitational force. So $mg - \frac{B^2 l^2 v}{R} = m\frac{dv}{dt}$

$\Rightarrow \frac{dv}{dt} = g - \alpha v$ where $\alpha = \frac{B^2 l^2}{mR}$

$\Rightarrow \frac{dv}{g - \alpha v} = dt \Rightarrow -\frac{1}{\alpha}\ln(g - \alpha v) = t + const. \Rightarrow g - \alpha v = Ae^{-\alpha t}$. At $t = 0$, $v = 0 \Rightarrow A = g$

$\Rightarrow v = \frac{g}{\alpha}\left(1 - e^{-\alpha t}\right)$

For small t; $\Rightarrow v \approx \frac{g}{\alpha}\left[1 - \left(1 - \alpha t + \frac{\alpha^2 t^2}{2} - ..\right)\right] \approx \frac{g}{\alpha}\left[\alpha t - \frac{\alpha^2 t^2}{2} + ..\right]$

Q74. A stationary magnetic dipole $\vec{m} = m\hat{k}$ is placed above an infinite surface $(z = 0)$ carrying a uniform surface current density $\vec{\kappa} = k\hat{i}$. The torque of the dipole is

(a) $\frac{\mu_0}{2} mk\hat{i}$

(b) $-\frac{\mu_0}{2} mk\hat{i}$

(c) $\frac{\mu_0}{2} mk\hat{j}$

(d) $-\frac{\mu_0}{2} mk\hat{j}$

Ans.:(a)

Solution:

Magnetic field due infinite sheet is $\vec{B} = -\frac{\mu_0 k}{2}\hat{j}$.

The torque of the dipole is $\vec{\tau} = \vec{m} \times \vec{B} = (m\hat{k}) \times \left(-\frac{\mu_0 k}{2}\hat{j}\right) = \frac{\mu_0 mk}{2}\hat{i}$

Part-B

Q24. In terms of a complete set of orthonormal basis kets $|n\rangle$,

$n = 0, \pm 1, \pm 2, \ldots$, the Hamiltonian is

$$H = \sum_n \left(E|n\rangle\langle n| + \in |n+1\rangle\langle n| + \in |n\rangle\langle n+1| \right)$$

where E and $\in$ are constants. The state $|\varphi\rangle = \sum_n e^{in\varphi}|n\rangle$ is an eigenstate with energy

(a) $E + \in \cos\varphi$ (b) $E - \in \cos\varphi$

(c) $E + 2\in \cos\varphi$ (d) $E - 2\in \cos\varphi$

Ans.: (c)

Solution:

Given $|\phi\rangle$ is an eigenstate of H. $\therefore H|\phi\rangle = \lambda|\phi\rangle$, where λ is the eigenvalue

Now $H|\phi\rangle = \sum_n \left(E|n\rangle\langle n| + \in |n+1\rangle\langle n| + \in |n\rangle\langle n+1| \right)|\phi\rangle$

$$= \sum_n \left(E|n\rangle\langle n| + \in |n+1\rangle\langle n| + \in |n\rangle\langle n+1| \right) \sum_m e^{im\phi}|m\rangle$$

$$= \sum_n \sum_m E|n\rangle\langle n|m\rangle e^{im\phi} + \sum_n \sum_m \in |n+1\rangle\langle n|m\rangle e^{im\phi} + \sum_n \sum_m \in |n\rangle\langle n+1|m\rangle e^{im\phi}$$

Since $\langle n|m\rangle = \delta_{n,m} = 1$; when $m = n$ and $\langle n+1|m\rangle = \delta_{n+1,m} = 1$; when $m = n+1$

$$\therefore \; H|\phi\rangle = \sum_n \left(E|n\rangle e^{in\phi} + \in |n+1\rangle e^{in\phi} + \in |n\rangle e^{i(n+1)\phi} \right)$$

Now put $n = n-1$ in second term

$$\therefore \; H|\phi\rangle = \sum_n \left(E|n\rangle e^{in\phi} + \in |n\rangle e^{i(n-1)\phi} + \in |n\rangle e^{i(n+1)\phi} \right) = \sum_n \left(E + \in \left(e^{i\phi} + e^{-i\phi} \right) \right) e^{in\phi}|n\rangle$$

$$= \left(E + 2\in \cos\phi \right) \sum_n e^{in\phi}|n\rangle = \lambda|\phi\rangle$$

Thus, eigen value is $\lambda = E + 2\in \cos\phi$. Therefore option (c) is correct.

Q27. The momentum space representation of the Schrodinger equation of a particle in a potential $V(\vec{r})$ is $\left(|\vec{p}|^2 + \beta\left(\nabla_p^2\right)^2 \right)\psi(\vec{p},t) = i\hbar\frac{\partial}{\partial t}\psi(\vec{p},t)$, where $\left(\nabla_p\right)_i = \frac{\partial}{\partial p_i}$, and β is a constant. The potential is (in the following V_0 and a are constants)

(a) $V_0 e^{-r^2/a^2}$ (b) $V_0 e^{-r^4/a^4}$

(c) $V_0\left(\frac{r}{a}\right)^2$ (d) $V_0\left(\frac{r}{a}\right)^4$

Ans.: (d)

Solution:

The Schrodinger equation in momentum space is given as

$$\left(|p|^2 + \beta\left(\nabla_p^2\right)^2\right)\psi(p,t) = i\hbar\frac{\partial}{\partial t}\psi(p,t)$$

The first term $\left(|p|^2\right)$ represent the kinetic energy and second term $\left(\beta\left(\nabla_p^2\right)^2\right)$ represent the potential energy.

In momentum space, operator $\hat{x}$ is written as $\hat{x} = i\hbar\dfrac{\partial}{\partial p}$

In the three-dimension, $\hat{r} = i\hbar\nabla p$

Now, $r^2 = -\hbar^2\nabla_p^2$ and $r^4 = \hbar^4\left(\nabla_p^2\right)^2$

Thus $\left(\nabla_p^2\right)^2 \propto r^4$. Therefore, option (d) is correct answer.

Q34. If the expectation value of the momentum of a particle in one dimension is zero, then its (box-normalizable) wave function may be of the form

(a) $\sin kx$ (b) $e^{ikx}\sin kx$

(c) $e^{ikx}\cos kx$ (d) $\sin kx + e^{ikx}\cos kx$

Ans.: (a)

Solution:

Expectation value of momentum for real wave function is always zero. Thus the correct wave function is $\sin(kx)$

Q37. Consider the Hamiltonian $H = AI + B\sigma_x + C\sigma_y$, where A, B and C are positive constants, I is the 2 × 2 identity matrix and σ_x, σ_y are Pauli matrices. If the normalized eigenvector corresponding to its largest energy eigenvalue is $\frac{1}{\sqrt{2}}\begin{pmatrix}1\\ y\end{pmatrix}$, then y is

(a) $\dfrac{B+iC}{\sqrt{B^2+C^2}}$ (b) $\dfrac{A-iB}{\sqrt{A^2+B^2}}$

(c) $\dfrac{A-iC}{\sqrt{A^2+C^2}}$ (d) $\dfrac{B-iC}{\sqrt{B^2+C^2}}$

Ans.: (a)

Solution: Given $H = AI + B\sigma_x + C\sigma_y$

$$H = A\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} + B\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix} + C\begin{bmatrix}0 & -i\\i & 0\end{bmatrix} = \begin{bmatrix}A & B-iC\\B+iC & A\end{bmatrix}$$

The eigenvalues of H is obtained from $|H - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix}A-\lambda & B-iC\\B+iC & A-\lambda\end{vmatrix} = 0 \Rightarrow (A-\lambda)^2 - (B+iC)(B-iC) = 0$$

$$\Rightarrow (A-\lambda)^2 - (B^2+C^2) = 0 \Rightarrow A-\lambda = \pm\sqrt{B^2+C^2} \;\; \Rightarrow \lambda = A \pm \sqrt{B^2+C^2}$$

The largest eigenvalue is $\lambda_1 = A + \sqrt{B^2+C^2}$ for the eigen state $|\phi\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\y\end{pmatrix}$

$$\therefore \; H|\phi\rangle = \lambda_1|\phi\rangle$$

$$\begin{bmatrix}A & B-iC\\B+iC & A\end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix}1\\y\end{bmatrix} = \frac{\lambda_1}{\sqrt{2}}\begin{bmatrix}1\\y\end{bmatrix} \Rightarrow \begin{bmatrix}A + y(B-iC)\\B+iC+Ay\end{bmatrix} = \begin{bmatrix}A+\sqrt{B^2+C^2}\\Ay + y\sqrt{B^2+C^2}\end{bmatrix}$$

$$\Rightarrow A + y(B-iC) = A + \sqrt{B^2+C^2} \quad \Rightarrow y = \frac{\sqrt{B^2+C^2}}{B-iC}\times\frac{B+iC}{B+iC} = \frac{(B+iC)\sqrt{B^2+C^2}}{B^2+C^2}$$

Therefore, $y = \dfrac{B+iC}{\sqrt{B^2+C^2}}$. Thus correct option is (a).

Part-C

Q53. The energy/energies E of the bound state(s) of a particle of mass m in one dimension in the

potential $V(x)=\begin{cases}\infty, & x\le 0\\ -V_0, & 0<x<a\\ 0, & x\ge a\end{cases}$ (where $V_0>0$) is/are determined by

(a) $\cot^2\left(a\sqrt{\frac{2m(E+V_0)}{\hbar^2}}\right)=\frac{E-V_0}{E}$

(b) $\tan^2\left(a\sqrt{\frac{2m(E+V_0)}{\hbar^2}}\right)=-\frac{E}{E+V_0}$

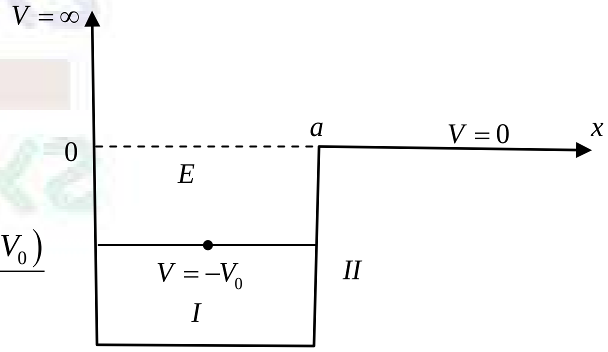
(c) $\cot^2\left(a\sqrt{\frac{2m(E+V_0)}{\hbar^2}}\right)=-\frac{E}{E+V_0}$

(d) $\tan^2\left(a\sqrt{\frac{2m(E+V_0)}{\hbar^2}}\right)=\frac{E-V_0}{E}$

Ans.: (c)

Solution:

Given $V(x)=\begin{cases}\infty & , x\le 0\\ -V_0 & , 0<x<a\\ 0 & , x\ge a\end{cases}$



The Schrodinger equation is region-I is

$$\frac{d^2\psi(x)}{dx^2}+\frac{2m}{\hbar^2}(E+V_0)\psi_I(x)=0 \qquad \text{Let } \alpha^2=\frac{2m(E+V_0)}{\hbar^2}$$

$$\Rightarrow \frac{d^2\psi_I(x)}{dx^2}+\alpha^2\psi_I(x)=0$$

The solution of this equation is $\psi_I(x)=A\cos(2x)+B\sin(2x)$

At $x=0,\ \psi_I(x)=0\Rightarrow A=0$. Therefore, $\psi_I(x)=B\sin(\alpha x)$ ....(1)

The Schrodinger equation is region II is

$$\frac{d^2\psi_{II}(x)}{dx^2}+\frac{2mE}{\hbar^2}\psi_{II}(x)=0 \ \Rightarrow \frac{d^2\psi_{II}(x)}{dx^2}-\beta^2\psi_{II}(x)=0 \quad \text{let } \beta^2=-\frac{2mE}{\hbar^2}$$

The solution is $\psi_{II}(x)=Ce^{+\beta x}+De^{-\beta x}$

Since, in region - II, $\psi_{II}(x)\to 0$ as $x\to+\infty$; $\therefore\ \psi_{II}(x)=De^{-\beta x}$ ....(2)

Using boundary condition

$\psi_I(a)=\psi_{II}(a)\Rightarrow B\sin(\alpha a)=De^{-\beta a}$ ....(3)

and $\frac{d\psi_I(a)}{dx}=\frac{d\psi_{II}(a)}{dx}\Rightarrow B\alpha\cos(\alpha a)=-\beta De^{-\beta a}$ ....(4)

Now divide (4) by (3), $\therefore\ \alpha\cot(\alpha a) = -\beta \Rightarrow \cot(\alpha a) = -\dfrac{\beta}{\alpha}$

Square on both sides $\quad \cot^2(\alpha a) = \dfrac{\beta^2}{\alpha^2} \Rightarrow \cot^2\left(\dfrac{\sqrt{2m(E+V_0)}}{\sqrt{\hbar^2}}\right) = \dfrac{-E}{E+V_0}$

Thus correct option is (c).

Q63. To first order in perturbation theory, the energy of the ground state of the Hamiltonian

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2x^2 + \frac{\hbar\omega}{\sqrt{512}}\exp\left[-\frac{m\omega}{\hbar}x^2\right]$$

(treating the third term of the Hamiltonian as a perturbation) is

(a) $\dfrac{15}{32}\hbar\omega$ (b) $\dfrac{17}{32}\hbar\omega$

(c) $\dfrac{19}{32}\hbar\omega$ (d) $\dfrac{21}{32}\hbar\omega$

Ans.: (b)

Solution:

According to Perturbation theorem, the first order correction in ground state energy is

$$E_0^{(1)} = \langle H'\rangle = \langle\psi_0|H'|\psi_0\rangle = \int_{-\infty}^{+\infty}\psi_0^* H'\psi_0 dx \quad \text{where } H' = \frac{\hbar\omega}{\sqrt{512}}\exp\left[-\frac{m\omega}{\hbar}x^2\right]$$

and ground state wave function of Harmonic oscillator is

$$\psi_0(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\exp\left(-\frac{m\omega}{2\hbar}x^2\right)$$

$$\therefore\ E_0^{(1)} = \left(\frac{m\omega}{\pi\hbar}\right)^{1/2}\frac{\hbar\omega}{\sqrt{512}}\int_{-\infty}^{+\infty} e^{-\frac{m\omega}{2\hbar}x^2}\cdot e^{-\frac{m\omega}{\hbar}x^2}\cdot e^{-\frac{m\omega}{2\hbar}x^2} d \;=\left(\frac{m\omega}{\pi\hbar}\right)^{1/2}\frac{\hbar\omega}{\sqrt{512}}\int_{-\infty}^{+\infty} e^{-\frac{2m\omega}{\hbar}x^2}dx$$

$$=\left(\frac{m\omega}{\pi\hbar}\right)^{1/2}\frac{2\hbar\omega}{\sqrt{512}}\int_{0}^{\infty} e^{-\frac{2m\omega}{\hbar}x^2}dx = \left(\frac{m\omega}{\pi\hbar}\right)^{1/2}\frac{2\hbar\omega}{\sqrt{512}}\cdot\frac{1}{2}\frac{\left\lceil\frac{1}{2}\right.}{\left(\frac{2m\omega}{\hbar}\right)^{1/2}}$$

$$=\left(\frac{m\omega}{\pi\hbar}\right)^{1/2}\cdot\frac{\hbar\omega}{\sqrt{512}}\cdot\frac{\sqrt{\pi}}{\left(\frac{2m\omega}{\hbar}\right)^{1/2}} = \frac{\hbar\omega}{32}$$

The ground state energy is $E_0 = \dfrac{\hbar\omega}{2} + \dfrac{\hbar\omega}{32} = \dfrac{17}{32}\hbar\omega$

Q66. The Hamiltonian for a spin-1/2 particle in a magnetic field $\vec{B} = B_0\hat{k}$ is given by $H = \lambda\vec{S}\cdot\vec{B}$, where $\vec{S}$ is its spin (in units of $\hbar$) and λ is a constant. If the average spins density is $\langle\vec{S}\rangle$ for an ensemble of such non-interacting particles, then $\frac{d}{dt}\langle S_x\rangle$

(a) $\frac{\lambda}{\hbar}B_0\langle S_x\rangle$

(b) $\frac{\lambda}{\hbar}B_0\langle S_y\rangle$

(c) $-\frac{\lambda}{\hbar}B_0\langle S_x\rangle$

(d) $-\frac{\lambda}{\hbar}B_0\langle S_y\rangle$

Ans.: (d)

Solution:

According to Ehrenfest theorem $\frac{d\langle A\rangle}{dt} = \frac{1}{i\hbar}\langle[A,H]\rangle + \left\langle\frac{\partial A}{\partial t}\right\rangle$

Given $H = \lambda\vec{S}\cdot\vec{B} = \lambda\left(S_x\hat{i} + S_y\hat{j} + S_z\hat{k}\right)\cdot\left(B_0\hat{k}\right) = \lambda B_0 S_z$

$\therefore \frac{d}{dt}\langle S_x\rangle = \frac{1}{i\hbar}\langle[S_x,H]\rangle + \left\langle\frac{\partial S_x}{\partial t}\right\rangle$ where $[S_x,H] = [S_x,\lambda B_0 S_z] = \lambda B_0[S_x,S_z] = \lambda B_0(-iSy)$

Now $\langle[S_x,H]\rangle = -i\lambda B_0\langle S_y\rangle$ and $\left\langle\frac{\partial S_x}{\partial t}\right\rangle = 0$

$\therefore \frac{d}{dt}\langle S_x\rangle = \frac{1}{i\hbar}\left(-i\lambda B_0\langle S_y\rangle\right) = -\frac{\lambda}{\hbar}B_0\langle S_y\rangle$. Thus correct option is (d)

Q69. At time $t = 0$, a particle is in the ground state of the Hamiltonian $H(t) = \frac{p^2}{2m} + \frac{1}{2}m\omega^2x^2 + \lambda x\sin\frac{\omega t}{2}$ where λ, ω and m are positive constants. To $O(\lambda^2)$, the probability that at $t = \frac{2\pi}{\omega}$, the particle would be in the first excited state of $H(t=0)$ is

(a) $\frac{9\lambda^2}{16m\hbar\omega^3}$

(b) $\frac{9\lambda^2}{8m\hbar\omega^3}$

(c) $\frac{16\lambda^2}{9m\hbar\omega^3}$

(d) $\frac{8\lambda^2}{9m\hbar\omega^3}$

Ans.: (d)

Solution:

According to time dependent perturbation theory, the probability of transition from initial to final state is

$$P_{if} = \frac{1}{\hbar^2}\left|\int_0^t e^{i\omega_{if}t} H'_{if}\, dt\right|^2 \qquad \text{where } H'_{if} = \langle 0|H'|1\rangle = \left\langle 0\left|\lambda x \sin\frac{\omega t}{2}\right|1\right\rangle = \lambda \sin\frac{\omega t}{2}\langle 0|x|1\rangle$$

$$\Rightarrow H'_{if} = \lambda \sin\left(\frac{\omega t}{2}\right)\cdot\sqrt{\frac{\hbar}{2m\omega}} \text{ and } \omega_{if} = \frac{E_f - E_i}{\hbar} = \frac{\hbar\omega}{\hbar} = \omega$$

$$\therefore P_{if} = \frac{1}{\hbar^2}\left|\int_0^{2\pi/\omega} \lambda \sin\left(\frac{\omega t}{2}\right)\sqrt{\frac{\hbar}{2m\omega}}\cdot e^{i\omega t} dt\right|^2 = \frac{\lambda^2}{\hbar^2}\left(\frac{\hbar}{2m\omega}\right)\left|\int_0^{2\pi/\omega} e^{i\omega t}\sin\left(\frac{\omega t}{2}\right)dt\right|^2 = \frac{\lambda^2}{\hbar^2}\left(\frac{\hbar}{2m\omega}\right)\left|-\frac{4}{3\omega}\right|^2$$

$$\Rightarrow P_{if} = \frac{\lambda^2}{\hbar^2}\cdot\frac{\hbar}{2m\omega}\times\frac{16}{9\omega^2} = \frac{8}{9}\frac{\lambda^2}{m\hbar\omega^3}.$$ Thus correct option is (d).

Part-B

Q31. If the average energy $\langle E\rangle_T$ of a quantum harmonic oscillator at a temperature T is such that $\langle E\rangle_T = 2\langle E\rangle_{T\to 0}$, then T satisfies

(a) $\coth\left(\frac{\hbar\omega}{k_BT}\right)=2$

(b) $\coth\left(\frac{\hbar\omega}{2k_BT}\right)=2$

(c) $\coth\left(\frac{\hbar\omega}{k_BT}\right)=4$

(d) $\coth\left(\frac{\hbar\omega}{2k_BT}\right)=4$

Ans.: (b)

Solution:

For Quantum Harmonic Oscillator $\varepsilon_n=\left(n+\frac{1}{2}\right)\hbar\omega$, $n=0,1,2,3,...$

$$Z=\sum_{n=0}^{\infty}e^{-\beta\varepsilon_n}=\sum_{n=0}^{\infty}e^{-\beta\left(n+\frac{1}{2}\right)\hbar\omega}=e^{-\frac{\beta\hbar\omega}{2}}+e^{-\frac{3}{2}\beta\hbar\omega}+e^{-\frac{5}{2}\beta\hbar\omega}+...=e^{-\frac{\beta\hbar\omega}{2}}\left[1+e^{-\beta\hbar\omega}+e^{-2\beta\hbar\omega}+...\right]$$

$$\Rightarrow Z=\frac{e^{-\frac{\beta\hbar\omega}{2}}}{1-e^{-\beta\hbar\omega}} \qquad \Rightarrow \ln Z=\frac{-\beta\hbar\omega}{2}-\ln\left(1-e^{-\beta\hbar\omega}\right)$$

Thus $\langle E\rangle=-\frac{\partial}{\partial\beta}\ln Z=\frac{\hbar\omega}{2}+\frac{-e^{-\beta\hbar\omega}}{1-e^{-\beta\hbar\omega}}(-\hbar\omega) \quad \Rightarrow \langle E\rangle_T=\left[\frac{1}{2}+\frac{1}{e^{\beta\hbar\omega}-1}\right]\hbar\omega$

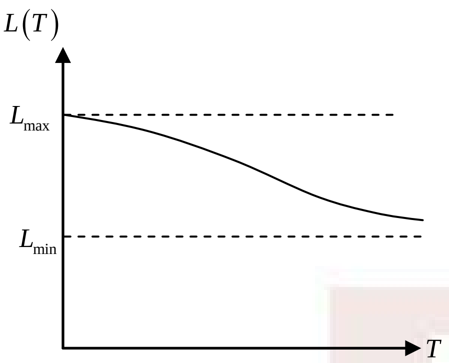
Now, as $T\to 0, \beta\to\infty, e^{\beta\hbar\omega}\to$ large; $\langle E\rangle_{T\to 0}=\frac{\hbar\omega}{2}$

$$\because \langle E\rangle_T=2\langle E\rangle_{T\to 0} \Rightarrow \left[\frac{1}{2}+\frac{1}{e^{\beta\hbar\omega}-1}\right]\hbar\omega=\hbar\omega \Rightarrow \frac{1}{2}+\frac{1}{e^{\beta\hbar\omega}-1}=1 \Rightarrow \frac{e^{\beta\hbar\omega}-1+2}{2\left(e^{\beta\hbar\omega}-1\right)}=1$$

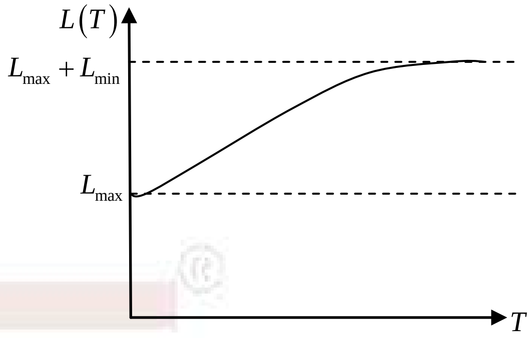
$$\Rightarrow \frac{e^{\beta\hbar\omega}+1}{e^{\beta\hbar\omega}-1}=2 \Rightarrow \frac{e^{\frac{\beta\hbar\omega}{2}}+e^{-\frac{\beta\hbar\omega}{2}}}{e^{\frac{\beta\hbar\omega}{2}}-e^{-\frac{\beta\hbar\omega}{2}}}=2 \Rightarrow \coth\left(\frac{\beta\hbar\omega}{2}\right)=2 \text{ or } \coth\left(\frac{\hbar\omega}{2k_BT}\right)=2$$

Q43. An elastic rod has a low energy state of length L_{max} and high energy state of length L_{min}. The best schematic representation of the temperature (T) dependence of the mean equilibrium length L(T) of the rod, is

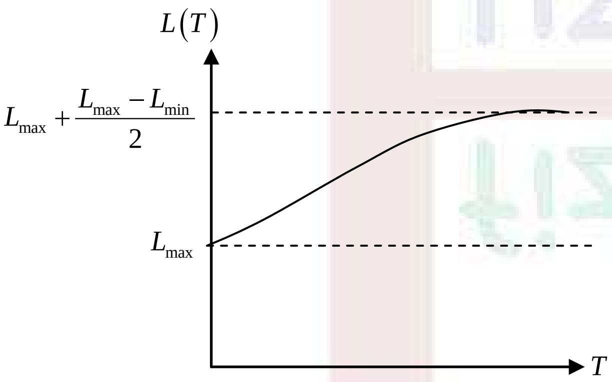
(a)



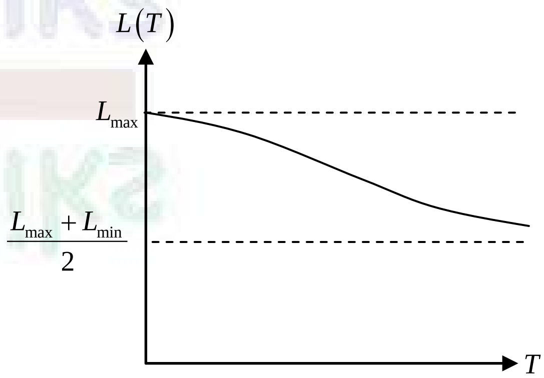
(b)



(c)



(d)



Ans.: (d)

Solution:

Let E_1 and E_2 represent the lowest and highest energy states of the elastic rod.

$E_1 = CL_{max}$ and $E_2 = CL_{min}$, where C is a constant of appropriate dimensions so that CL has dimensions of energy.

Let $P(E_1)$ and $P(E_2)$ are the probabilities of the rod to have states with energy E_1 and E_2, respectively. Then, $P(E_1)=\dfrac{e^{-\beta E_1}}{e^{-\beta E_1}+e^{-\beta E_2}}$, $P(E_2)=\dfrac{e^{-\beta E_2}}{e^{-\beta E_1}+e^{-\beta E_2}}$

$$\langle L\rangle = L_{max}P(E_1)+L_{min}P(E_2) \quad ....(1)$$

As $T\to\infty$, $\beta\to 0$, $e^{-\beta E_1}\approx e^0=1$, $\because \dfrac{E_1}{k_BT}\approx 0$ as E_1 is small and $T\to\infty$

$e^{-\beta E_2}\approx e^0=1$, $\because \dfrac{E_2}{k_BT}\approx 0$, E_2 is large but $T\to\infty$

$\therefore$ In this case $P(E_1)=P(E_2)=\dfrac{1}{1+1}=\dfrac{1}{2}$ and $\langle L\rangle=\dfrac{L_{\max}+L_{\min}}{2}$

In the other extreme, when $T\to 0, \beta=\dfrac{1}{k_B T}\to\infty$

$$\langle L\rangle=\frac{e^{-\beta E_1}}{e^{-\beta E_1}+e^{-\beta E_2}}L_{\max}+\frac{e^{-\beta E_2}}{e^{-\beta E_1}+e^{-\beta E_2}}L_{\min}=\frac{e^{-\beta E_1}L_{\max}+e^{-\beta E_2}L_{\min}}{e^{-\beta E_1}+e^{-\beta E_2}}$$

$$\langle L\rangle=\frac{e^{-\beta E_1}\left[L_{\max}+e^{-\beta(E_2-E_1)}\right]}{e^{-\beta E_1}\left[1+e^{-\beta(E_2-E_1)}\right]}=\frac{L_{\max}+e^{-\beta(E_2-E_1)}}{1+e^{-\beta(E_2-E_1)}}\approx L_{\max}, \qquad \beta\to\infty \text{ as } T\to 0$$

Q45. A thermally isolated container, filled with an ideal gas at temperature *T*, is divided by a partition, which is clamped initially, as shown in the figure below.

The partition does not allow the gas in the two parts to mix. It is subsequently released and allowed to move freely with negligible friction. The final pressure at equilibrium is

(a) 5*P*/3 (b) 5*P*/4

(c) 3*P*/5 (d) 4*P*/5

Ans. :(a)

Solution:

$\because$ Vessel is isolated, $\Delta Q=0$. Since both partitions are at same temperature, we have for left and right part,

$PV=n_1RT$ ....(i)

$4PV=n_2RT$ ....(ii)

Here, we considered number of moles to be different in two parts.

Adding (i) and (ii)

$$5PV=(n_1+n_2)RT\Rightarrow(n_1+n_2)=\frac{5PV}{RT} \qquad \text{....(iii)}$$

Now, after mixing, let P_f be the final pressure, then

$$P_f(3V)=(n_1+n_2)RT \quad \Rightarrow P_f=\frac{(n_1+n_2)RT}{3V}=\frac{5PV}{3V}\frac{RT}{RT}$$

$P_f=\dfrac{5}{3}P$, $\therefore$ (a) is correct.

Part-C

Q49. A system of N non-interacting particles in one-dimension, each of which is in a potential $V(x)=gx^6$ where $g>0$ is a constant and x denotes the displacement of the particle from its equilibrium position. In thermal equilibrium, the heat capacity at constant volume is

(a) $\frac{7}{6}Nk_B$ (b) $\frac{4}{3}Nk_B$ (c) $\frac{3}{2}Nk_B$ (d) $\frac{2}{3}Nk_B$

Ans.: (d)

Solution: $V(x)=gx^6$; $E=\frac{p_x^2}{2m}+gx^6 \quad \Rightarrow \langle E\rangle=\left\langle\frac{p_x^2}{2m}\right\rangle+g\langle x^6\rangle$.

For $V(x)=ax^n$, a is a constant; $\langle V\rangle=\frac{k_BT}{n}$

$$\therefore \ \langle E\rangle=\frac{k_BT}{2}+\frac{k_BT}{6}=\frac{3k_BT+k_BT}{6}=\frac{4}{6}k_BT \ \Rightarrow \langle E\rangle=\frac{2}{3}k_BT$$

For N such non-interacting particles $U=N\langle E\rangle=\frac{2}{3}Nk_BT \ \Rightarrow C_V=\left(\frac{dU}{dT}\right)_V=\frac{2}{3}Nk_B$

Q55. The energy levels of a system, which is in equilibrium at temperature $T=1/(k_B\beta)$, are 0, $\in$ and $2\in$. If two identical bosons occupy these energy levels, the probability of the total energy being $3\in$, is

(a) $\frac{e^{-3\beta\in}}{1+e^{-\beta\in}+e^{-2\beta\in}+e^{-3\beta\in}+e^{-4\beta\in}}$ (b) $\frac{e^{-3\beta\in}}{1+2e^{-\beta\in}+2e^{-2\beta\in}+e^{-3\beta\in}+e^{-4\beta\in}}$

(c) $\frac{e^{-3\beta\in}}{e^{-\beta\in}+2e^{-2\beta\in}+e^{-3\beta\in}+e^{-4\beta\in}}$ (d) $\frac{e^{-3\beta\in}}{1+e^{-\beta\in}+2e^{-2\beta\in}+e^{-3\beta\in}+e^{-4\beta\in}}$

Ans.: (d)

Solution:

Two Bosons can be distributed in three energy levels as below

		00		0	0	2ε
	00		0	0	0	ε
00			0			0
0	2ε	4ε	ε	2ε	3ε	

There are six microstates, out of which only one has energy 3ε . Corresponding probability is

$$P(3\varepsilon)=\frac{e^{-3\beta\varepsilon}}{Z}=\frac{e^{-3\beta\varepsilon}}{1+e^{-\beta\varepsilon}+2e^{-\beta\varepsilon}+e^{-3\beta\varepsilon}+e^{-4\beta\varepsilon}}$$

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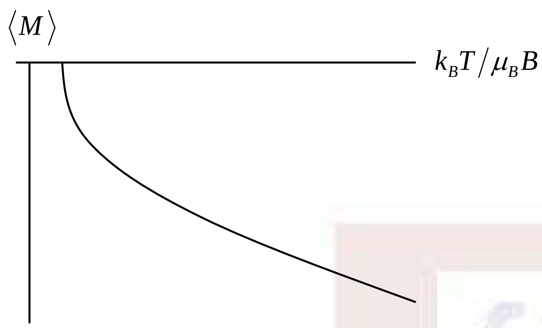
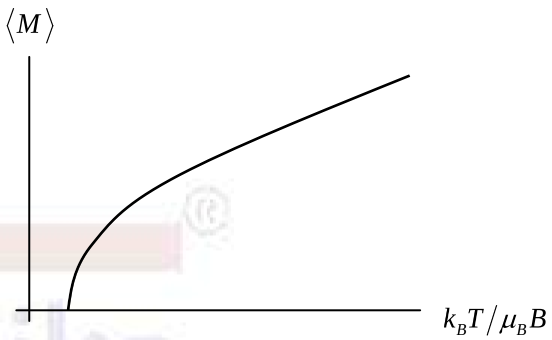
Q65. A paramagnetic salt with magnetic moment per ion $\mu_{\pm} = \pm\mu_B$ (where μ_B is the Bohr magneton) is in thermal equilibrium at temperature T in a constant magnetic field B. The average magnetic moment $\langle M \rangle$, as a function of $\dfrac{k_B T}{\mu_B B}$, is best represented by

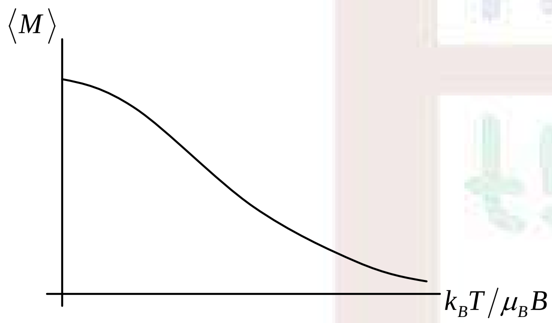
(a)



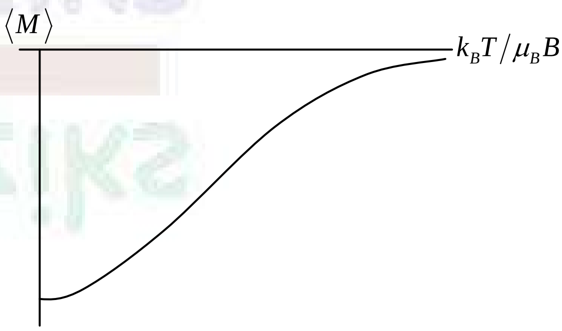
(b)



(c)



(d)



Ans.: (c)

Solution: Each spin with magnetic moment $\mu = \mu_B$ has two possible spin orientations. Corresponding interaction energies in external constant field B are $+\mu_B B$ and $-\mu_B B$. Let us denote these energies as $+\varepsilon$ and $-\varepsilon$, respectively.

$\therefore$ The partition function for single spin system is $Q_1(\beta) = e^{\beta\varepsilon} + e^{-\beta\varepsilon}$ ....(1)

$$Q_N(\beta) = [Q_1]^N = \left[e^{\beta\varepsilon} + e^{-\beta\varepsilon}\right]^N \Rightarrow Q_N(\beta) = [2\cosh\beta\varepsilon]^N \quad \text{....(2)}$$

The Helmholtz-free energy is $A = -k_B T \ln Q_N(\beta) = -Nk_B T \ln\left\{2\cosh\dfrac{\varepsilon}{kT}\right\}$ ....(3)

The magnetization is obtained as below:

$$\langle M \rangle = \frac{N}{\beta}\frac{\partial}{\partial B}\ln Q_1 = Nk_B T\frac{\partial}{\partial B}\ln Q_1 = k_B T\frac{\partial}{\partial B}\ln Q_1^N = \frac{\partial}{\partial B}k_B T\ln Q_1^N = -\left(\frac{\partial A}{\partial B}\right)_T$$

$$\langle M \rangle = -\left(\frac{\partial A}{\partial B}\right)_T = Nk_B T\frac{\partial}{\partial B}\left[\ln 2\cosh\frac{\mu_B B}{k_B T}\right] = Nk_B T\frac{1}{2\cosh\left(\dfrac{\mu_B B}{k_B T}\right)}\times 2\sinh\left(\frac{\mu_B B}{k_B T}\right)\left(\frac{\mu_B}{k_B T}\right)$$

$$\Rightarrow \langle M \rangle = N\mu_B \tanh\left(\frac{\mu_B B}{k_B T}\right) = N\mu_B \tanh\left(\frac{\varepsilon}{k_B T}\right) \qquad ....(4)$$

Let us approximate $\tanh\left(\frac{\varepsilon}{k_B T}\right)$ as below

(i) High T and low B so that $\beta\varepsilon \ll 1$

$$\tanh(\beta E) \approx \beta\varepsilon - \frac{1}{3}(\beta\varepsilon)^3 + ... \,, \quad \because \tanh x \approx x - \frac{x^3}{3}..., \; x \ll 1 \qquad ....(5)$$

$M = N\mu_B \tanh(\beta\varepsilon) \approx N\mu_B(\beta\varepsilon) \to 0$ as $\beta\varepsilon \ll 1$ i.e. a state of complete randomization

(ii) Low T and high B, i.e. $\beta\varepsilon \gg 1$; $\quad \tanh(\beta\varepsilon) = \frac{e^{\beta\varepsilon} - e^{-\beta\varepsilon}}{e^{\beta\varepsilon} + e^{-\beta\varepsilon}} \simeq \frac{e^{\beta\varepsilon}}{e^{\beta\varepsilon}} \simeq 1$ ....(6)

$M \approx N\mu_B$, $\because \tanh(\beta\varepsilon) \approx 1$ i.e. system is completely magnetized.

$\therefore$ (c) is correct.

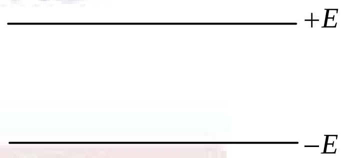
Q72. The energies of a two-level system are $\pm E$. Consider an ensemble of such non-interacting systems at a temperature *T*. At low temperatures, the leading term in the specific heat depends on *T* as

(a) $\frac{1}{T^2} e^{-E/k_B T}$ (b) $\frac{1}{T^2} e^{-2E/k_B T}$

(c) $T^2 e^{-E/k_B T}$ (d) $T^2 e^{-2E/k_B T}$

Ans.: (b)

Solution: The partition function is $Z = e^{\beta E} + e^{-\beta E} = 2\cosh(\beta E)$



$$\langle E \rangle = -\frac{\partial(\ln Z)}{\partial\beta} = -\frac{\partial}{\partial\beta}\ln\cosh(\beta E) = -\frac{1}{\cosh(\beta E)}\sinh(\beta E) \times E$$

$$\langle E \rangle = -E\tanh(\beta E)$$

$$C_V = \frac{\partial\langle E \rangle}{dT} = \frac{-d}{dT}\left[E\tanh(\beta E)\right] = -E\sec h^2(\beta E)\frac{d}{dT}\left(\frac{E}{k_B T}\right) = -E\sec h^2(\beta E) \times \left(-\frac{E}{k_B T^2}\right)$$

$$C_V = \frac{E^2}{k_B T^2}\sec h^2(\beta E)$$

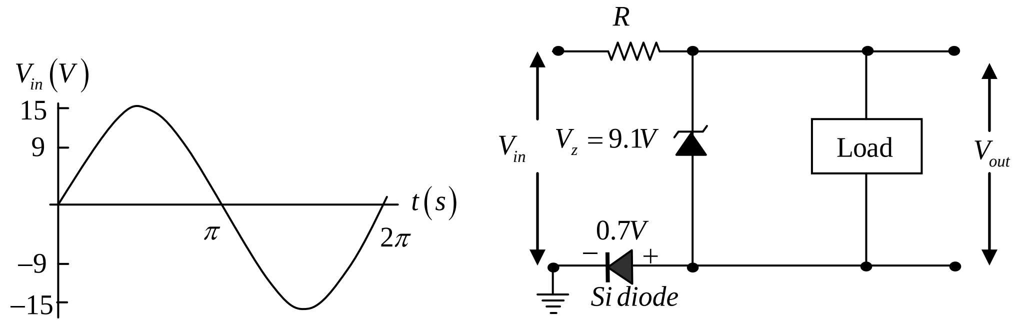
Now, $\sec h^2(\beta E) = \frac{1}{\cosh^2(\beta E)} = \frac{4}{\left(e^{\beta E} + e^{-\beta E}\right)^2}$

when $T \to 0, \beta \to$ high and $e^{-\beta E}$ is low, therefore $e^{\beta E} + e^{-\beta E} \approx e^{\beta E}$, $\sec h^2(\beta E) \approx 4e^{-2\beta E}$

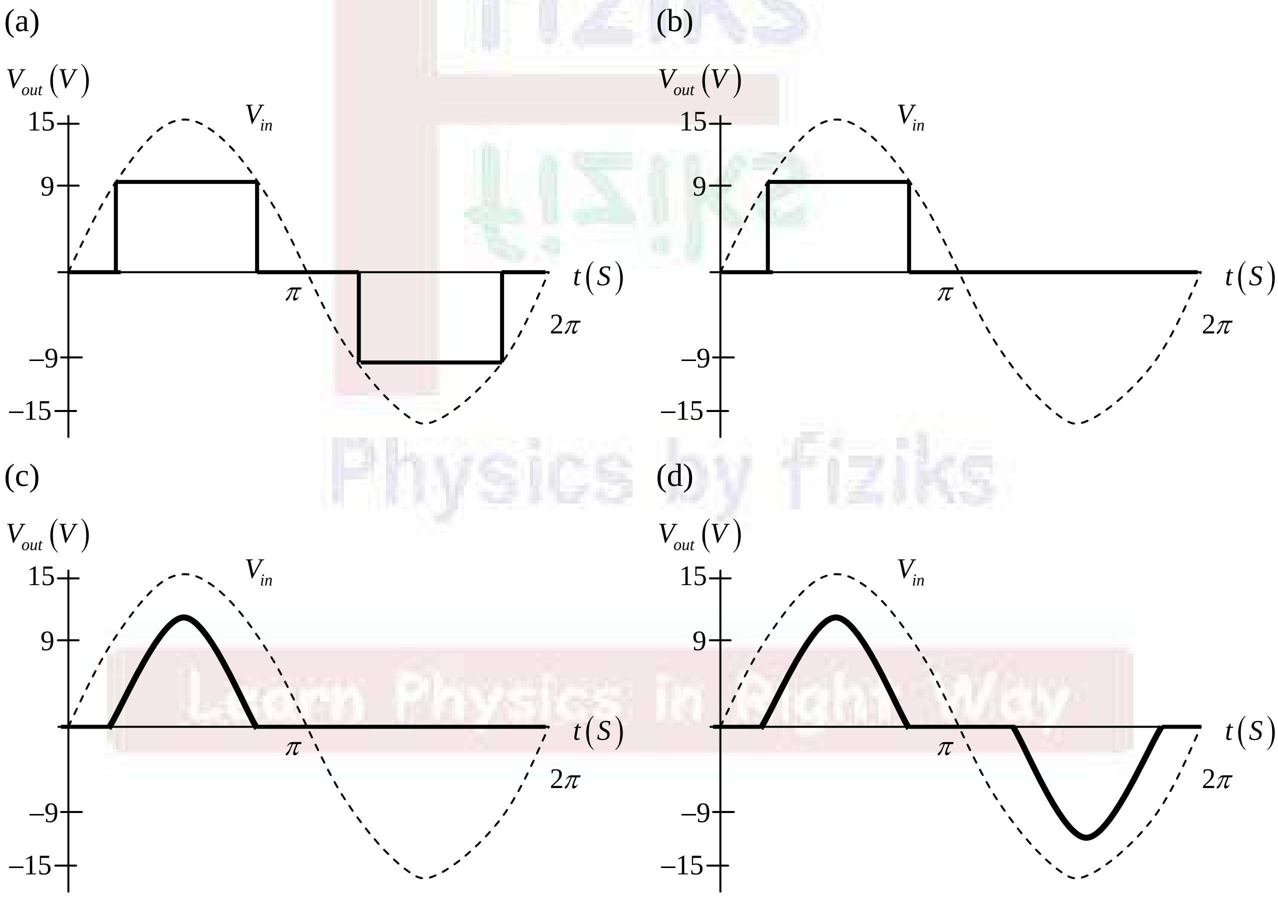
$$\therefore \; C_V \propto \frac{1}{T^2} e^{-2\beta E}$$

Part-B

Q21. A high impedance load (network) is connected in the circuit as shown below.



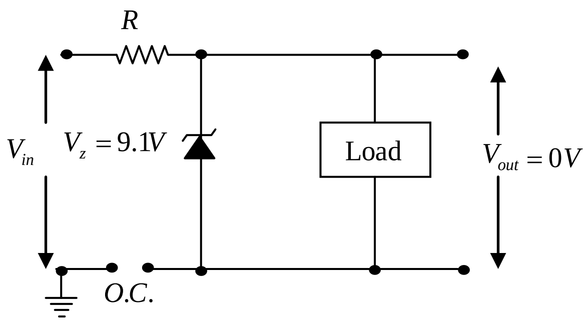
The forward voltage drop for silicon diode is 0.7 V and the Zener voltage is 9.10 V. If the input voltage (V_{in}) is sine wave with an amplitude of 15 V (as shown in the figure above), which of the following waveform qualitatively describes the output voltage (V_{out}) across the load?



Ans.:(c)

Solution.:

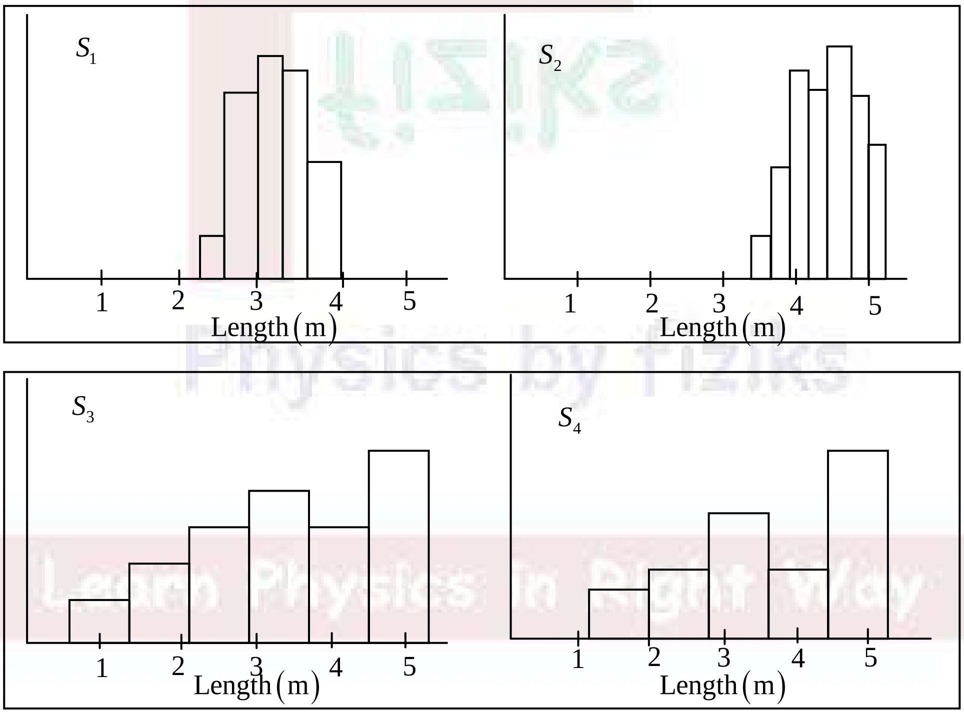
If $V_{in} < (9.1V + 0.7V)$



If $V_{in} > (9.1V + 0.7V)$ then zener is in breakdown rgion and Si-diode is forward bias and output can not exceed $9.1V$.

If V_{in} is negative then zener is forward bias and Si-diode is reverse bias so output is zero.

Q33. Four students (S_1, S_2, S_3 and S_4) make multiple measurements on the length of a table. The binned data are plotted as histograms in the following figures.



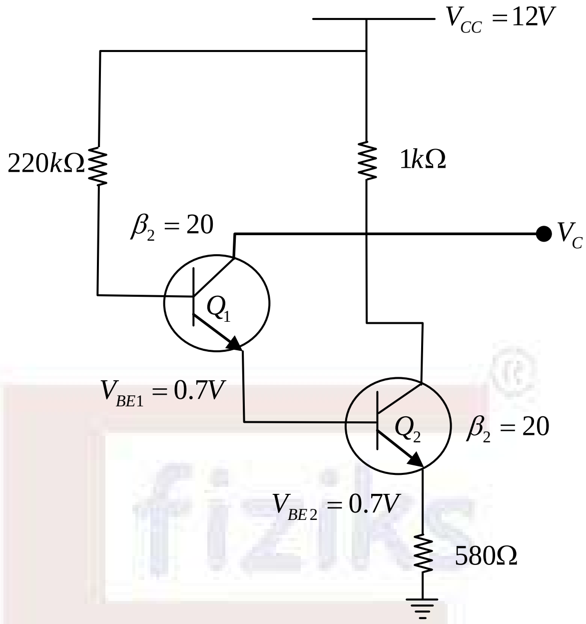
If the length of the table, specified by the manufacturer, is 3*m*, the student whose measurements have the minimum systematic error, is

(a) S_2 (b) S_1

(c) S_4 (d) S_3

Ans.: (b)

Q38. The figure below shows a circuit with two transistors, Q_1 and Q_2, having current gains β_1 and β_2 respectively.



The collector voltage V_C will be closest to

(a) 0.9 V (b) 2.2 V

(c) 2.9 V (d) 4.2 V

Ans.: (b)

Solution.:

$$I_{B1} = \frac{V_{CC} - V_{BE1} - V_{BE2}}{R_B + (\beta_D + 1) R_E}$$

$$= \frac{12V - 0.7V - 0.7V}{220 \times 10^3 + (400 + 1) 580} = 23.42 \mu A$$

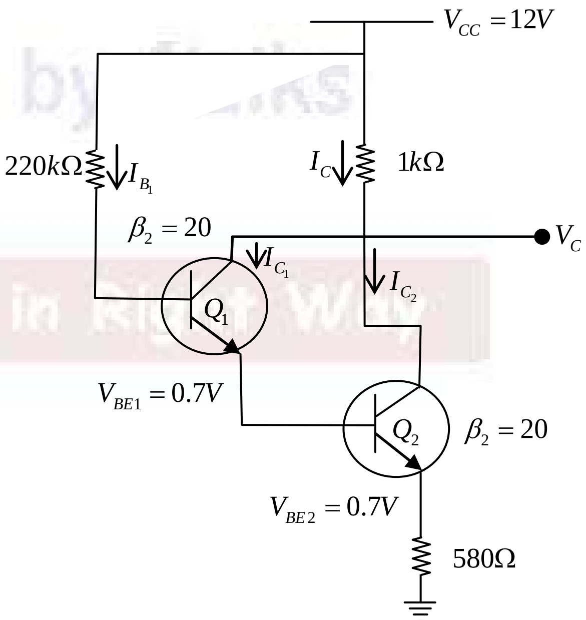
$$I_{C1} = \beta I_{B1} = 20 \times 23.42 \times 10^{-6} A = 0.47\, mA$$

$$I_{C2} = \beta_D I_{B2} = 400 \times 23.42 \times 10^{-6} A = 9.37\, mA$$

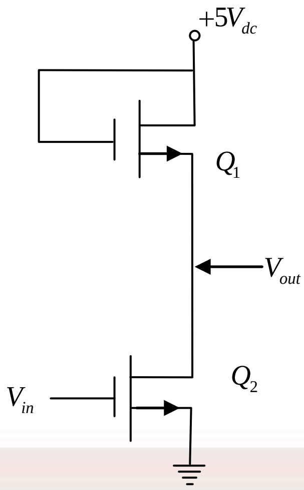
$$I_C = I_{C1} + I_{C2} = 0.47\, mA + 9.37\, mA = 9.84\, mA$$

$$V_C = V_{CC} - I_C R_C$$

$$= 12V - 9.84 \times 10^{-3} \times 1 \times 10^3 = 2.16\, V \approx 2.2\, V$$



Q39. The circuit containing two *n*-channel MOSFETs shown below, works as

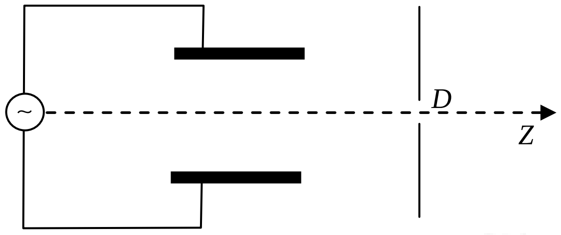


(a) a buffer
(b) a non-inverting amplifier
(c) an inverter
(d) a rectifier

Ans.: (c)

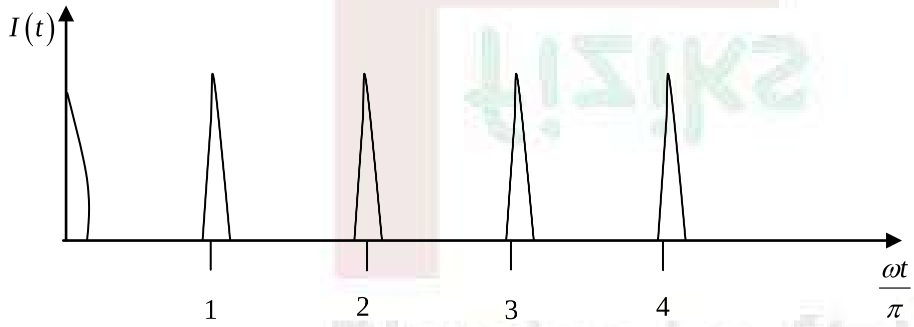
Part-C

Q56. A high frequency voltage signal $V_i = V_m \sin \omega t$ is applied to a parallel plate deflector as shown in the figure.

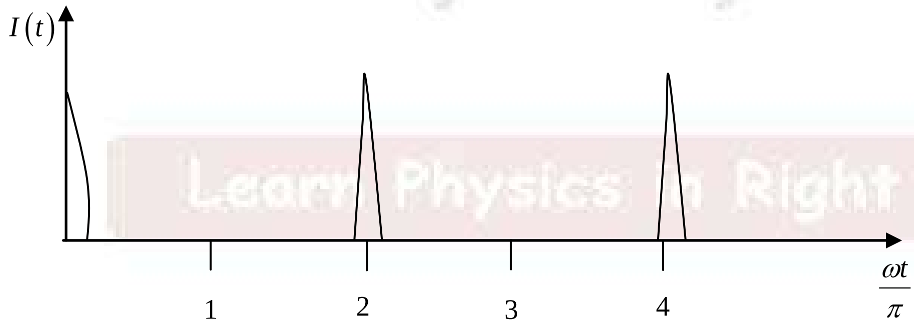


An electron beam is passing through the deflector along the central line. The best qualitative representation of the intensity $I(t)$ of the beam after it goes through the narrow circular aperture *D*, is

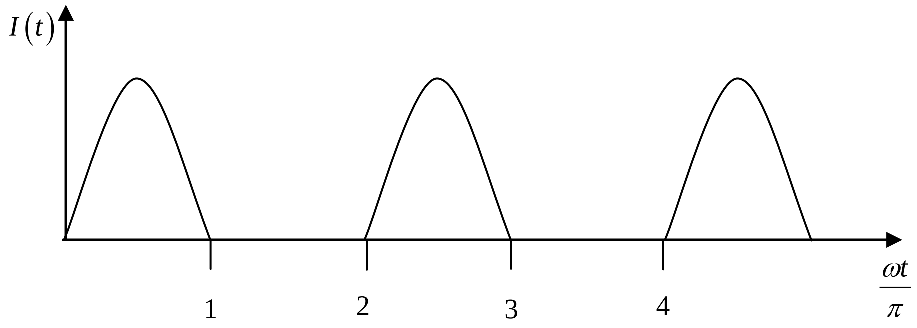
(a)



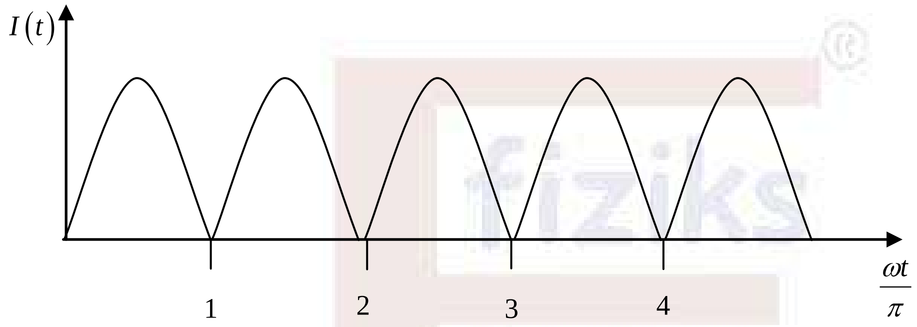
(b)



(c)



(d)



Ans.: (a)

Q57. An amplifier with a voltage gain of 40 dB without feedback is used in an electronic circuit. A negative feedback with a fraction 1/40 is connected to the input of this amplifier. The net gain of the amplifier in the circuit is closest to

(a) 40 dB (b) 37 dB (c) 29 dB (d) 20 dB

Ans.:(c)

Solution.:

$$A_F = \frac{A}{1+AB} = \frac{100}{1+100\times 1/40} = \frac{100}{3.5} = 28.57 \qquad \because 40\,dB = 20\log_{10} A \Rightarrow A = 100$$

Gain in $dB = 20\log_{10} A_F = 20\log_{10} 28.57 = 29\,dB$

Q64. A receiver operating at 27°C has an input resistance of 100Ω. The input thermal noise voltage for this receiver with a bandwidth of 100 kHz is closest to

(a) 0.4 nV (b) 0.6 pV

(c) 40 mV (d) 0.4 μV

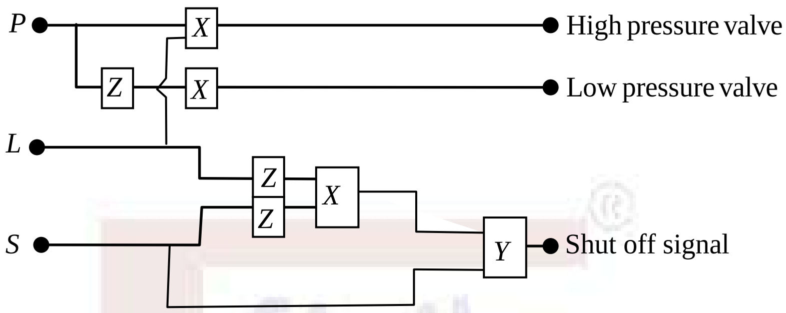
Ans.:(d)

Solution.:

$$\overline{V_n^2} = 4k_B TR\Delta f = 4\times 13.8\times 10^{-23}\times 300K\times 100\times 100\times 10^3 = 1.656\times 10^{-13}$$

$$V_n = \sqrt{1.656\times 10^{-13}} = 0.41\mu V$$

Q75. A liquid oxygen cylinder system is fitted with a level-sensor (L) and a pressure-sensor (P), as shown in the figure below. The output of L and P are set to logic high $(S = 1)$ when the measured values exceed the respective preset threshold values. The system can be shut off either by an operator by setting the input S to high, or when the level of oxygen in the tank falls below the threshold value



The logic gates X, Y and Z, respectively, are

(a) OR, AND and NOT

(b) AND, OR and NOT

(c) NAND, OR and NOT

(d) NOR, AND and NOT

Ans.:(b)

Part-C

Q51. The electronic configuration of ^{12}C is $1s^2 2s^2 2p^2$. Including LS coupling, the correct ordering of its energies is

(a) $E\left({}^3P_2\right) < E\left({}^3P_1\right) < E\left({}^3P_0\right) < E\left({}^1D_2\right)$ (b) $E\left({}^3P_0\right) < E\left({}^3P_1\right) < E\left({}^3P_2\right) < E\left({}^1D_2\right)$

(c) $E\left({}^1D_2\right) < E\left({}^3P_2\right) < E\left({}^3P_1\right) < E\left({}^3P_0\right)$ (d) $E\left({}^3P_1\right) < E\left({}^3P_0\right) < E\left({}^3P_2\right) < E\left({}^1D_2\right)$

Ans.: (b)

Solution: $l_1 = 1,\ l_2 = 2 \Rightarrow L = 0,1,2$ and $s_1 = \frac{1}{2},\ s_2 = \frac{1}{2} \Rightarrow S = 0,1$

Since p^2 is equivalent electron system, only odd L can combine with odd S to give J and even L can combine with even S to give J

$S=0$	$L=0$	$J=0$	${}^1S_0$
$S=0$	$L=2$	$J=2$	${}^1D_2$
$S=1$	$L=1$	$J=0,1,2$	${}^3P_{0,1,2}$

The energy will be in the order: $E\left({}^3P_0\right) < E\left({}^3P_1\right) < E\left({}^3P_2\right) < E\left({}^1D_2\right)$

Q58. The Raman rotational-vibrational spectrum of nitrogen molecules is observed using an incident radiation of wavenumber 12500 cm^{-1}. In the first shift band, the wavenumbers of the observed lines (in cm^{-1}) are 10150, 10158, 10170, 10182 and 10190. The values of vibrational frequency and rotational constant (in cm^{-1}), respectively, are

(a) 2330 and 2 (b) 2350 and 2

(c) 2350 and 3 (d) 2330 and 3

Ans.: (a)

Solution: Raman rotational-vibrational energy can be expressed as

$$\varepsilon_{v,J} = \omega_e\left(v + \frac{1}{2}\right) - \omega_e x_2\left(v + \frac{1}{2}\right)^2 + BJ\left(J+1\right)$$

From given data, the central line is $10170\,cm^{-1}$ which can be considered as the mid point the series which corresponds to $\Delta v = \pm 1$ and $\Delta J = 0$ i.e. purely vibrational. Hence, we can calculate the vibrational frequency in wavenumber and is given as difference between the two

i.e., $\bar{v} = \left(12500 - 10170\right)cm^{-1} = 2330\,cm^{-1}$

The change in wave number going on either side from the central line to the next line is due to rotation and hence rotational constant can be calculated. From the given data

$$6B = 12\,cm^{-1} \Rightarrow B = 2\,cm^{-1}$$

Q62. In the absorption spectrum of H-atom, the frequency of transition from the ground state to the first excited state is ν_H. The corresponding frequency for a bound state of a positively charged muon (μ^+) and an electron is ν_μ. Using $m_\mu = 10^{-28}$ kg, $m_e = 10^{-30}$ kg and $m_p \gg m_e, m_\mu$, the value of $(\nu_\mu - \nu_H)/\nu_H$ is

(a) 0.001
(b) – 0.001
(c) – 0.01
(d) 0.01

Ans.: (c)

Solution:

The absorption frequency for the transition in H-atom is given by:

$$\nu_H = \frac{c}{\lambda} = R_H c\left(\frac{1}{1} - \frac{1}{2^2}\right) = \frac{3R_H c}{4}$$

The absorption frequency for the transition in Muon-atom is given by:

$$\nu_\mu = \frac{c}{\lambda} = R_H c\left(\frac{1}{1} - \frac{1}{2^2}\right) = \frac{3R_\mu c}{4}$$

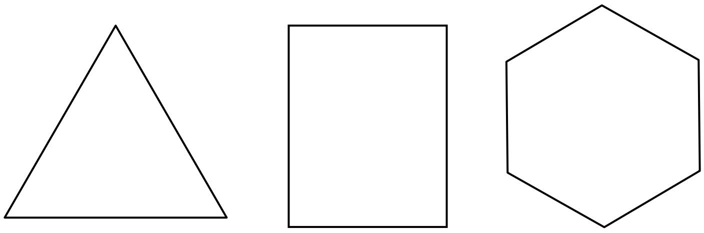
$R_\mu = \mu R_\infty$ and $\mu = \dfrac{m_\mu m_e}{m_\mu + m_e} = \dfrac{10^{-28} m_e}{10^{-28} + 10^{-30}} = \dfrac{10^{-28} m_e}{10^{-28}\left(1 + 10^{-2}\right)} = \dfrac{m_e}{1.01}$

and $R_H = m_e R_\infty$ since $\mu_H = \dfrac{m_p m_e}{m_p + m_e} \approx \dfrac{m_\mu m_e}{m_\mu} \approx m_e$

Thus $\dfrac{\nu_\mu - \nu_H}{\nu_H} = \dfrac{R_\mu - R_H}{R_H} = \dfrac{\frac{m_e}{1.01}R_\infty - m_e R_\infty}{m_e R_\infty} = \dfrac{\frac{1}{1.01} - 1}{1} \approx -0.01$

Part-C

Q50. The Figures (i), (ii) and (iii) below represent an equilateral triangle, a rectangle and a regular hexagon, respectively.



(i) (ii) (iii)

Which of these can be primitive unit cells of a Bravais lattice in two dimensions?

(a) only (i) and (iii) but not (ii)
(b) only (i) and (ii) but not (iii)
(c) only (ii) and (iii) but not (i)
(d) All of them

Ans.: (c)

Solution:

A Bravais lattice must have a translational symmetry. It is not possible to make Bravais lattice with triangle cell, as it does not have translational symmetry.

Part-C

Q47. The elastic scattering process $\pi^- p \to \pi^- p$ may be treated as a hard-sphere scattering. The mass of π^-, $m_\pi \simeq \frac{1}{6} m_p$, where $m_p \simeq 938\,\mathrm{MeV}/\mathrm{c}^2$ is the mass of the proton. The total scattering cross-section is closet to

(a) 0.01 milli-barn (b) 1 milli-barn

(c) 0.1 barn (d) 10 barn

Ans.: (c)

Solution: Scattering cross section in case of two hard sphere scattering case can be written as

$$\sigma = \pi\left(R_1 + R_2\right)^2 \simeq \pi\left(2R\right)^2 \simeq 4\pi R^2 = 4\times 3.14\times\left(1.2\times 10^{-15}\right)^2$$

$$\Rightarrow \sigma = 0.18\times 10^{-28}\, m^2 \quad \Rightarrow \sigma = 0.18\,\text{barn}$$

Q61. The tensor component of the nuclear force may be inferred from the fact that deuteron nucleus ${}_1^2H$

(a) has only one bound state with total spin $S = 1$

(b) has a non-zero electric quadrupole moment in its ground state

(c) is stable while triton ${}_1^3H$ is unstable

(d) is the only two nucleon bound state

Ans.: (b)

Solution: The ground state wave function of deuteron nucleus is

$$\psi\left({}_1^2D\right) = a\psi_1\left({}^3S_1\right) + b\psi_2\left({}^3D_1\right)$$

Non-zero electric quadrupole moment is due to non-symmetrical part $\psi_2\left({}^3D_1\right)$. This $\psi_2\left({}^3D_1\right)$ is also responsible for non-central part of nuclear force.

Q73. Thermal neutrons may be detected most efficiently by a

(a) ^{6}Li loaded plastic scintillator (b) Geiger-Muller counter

(c) inorganic scintillator CaF_2 (d) silicon detector

Ans.: (a)

Solution: If a scintillation counter has a phosphor made of LiI, the incident neutrons can interact with ${}_3^6Li$ to form ${}_2^4He$ and ${}_1^3H$ ions as given by the following exoergic reaction:

$${}_0^1n + {}_3^6Li \to {}_2^4He + {}_1^3H$$

The incident neutrons produce ${}_2^4He$ and ${}_1^3H$ which are ionizing particles and these produce an electrical pulse in scintillation counter.

Part-A

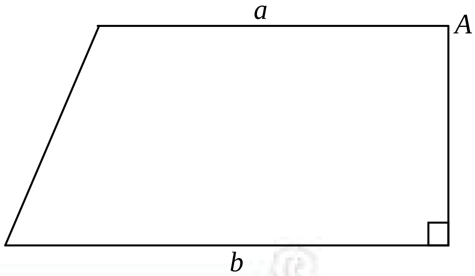
Q1. At what horizontal distance from A should a vertical line be drawn so as to divide the area of the trapezium shown in the figure into two equal parts? (a and b are lengths of the parallel sides.)

(a) $(a+b)/4$

(b) $(a+b)/3$

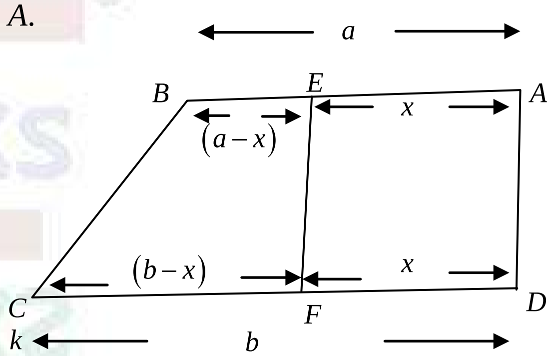
(c) $(a+b)/2$

(d) $(2a+b)/2$



Ans.: (a)

Solution: Let vertical line drawn be at a distance x from A.



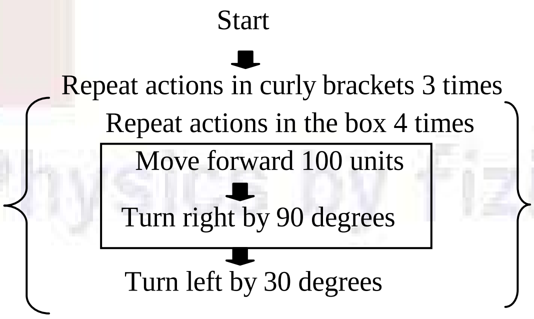
$EA = x = FD$; $BE = a - x$; $CF = b - x$

Also, Area $(\square EBCF)$ = area $(AEFD)$

$$\frac{1}{2}\times EF\times(a-x+b-x) = EF\times x$$

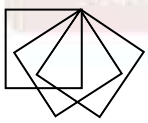
$$\Rightarrow a+b-2x = 2x \Rightarrow 4x = a+b \Rightarrow x = \frac{a+b}{4}$$

Q2. Starting from the top of a page and pointing downward, an ant moves according to the following commands

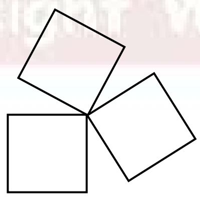


Of the following paths

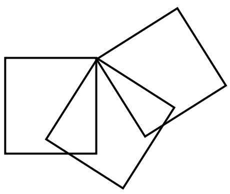
(A)



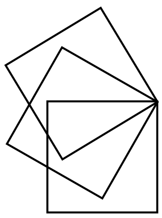
(B)



(C)



(D)



Which is the correct path of the ant?

(a) A (b) B

(c) C (d) D

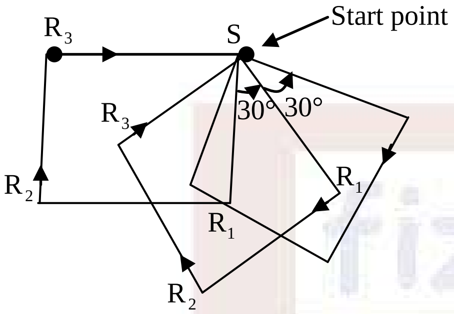
Ans.: (a)

Solution:

Move forward 100 units
Turn right by 90 degrees $\Rightarrow$ A square $\downarrow$ Then left by 30° } Action 1

$\downarrow$ Square $\downarrow$ Left by 30° } Action 2

$\downarrow$ Square $\downarrow$ Left by 30° } Action 3

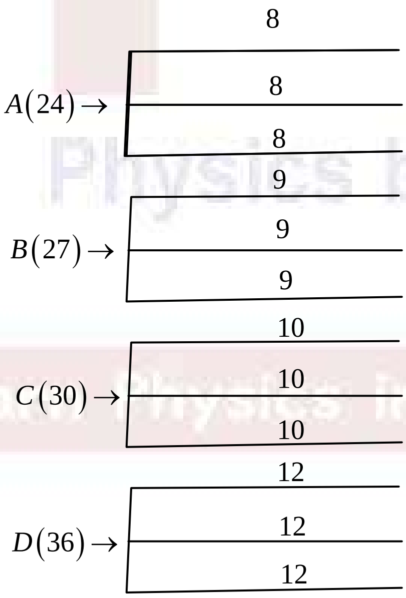


Q3. Sections *A*, *B*, *C* and *D* of a class have 24, 27, 30 and 36 students, respectively. One section has boys and girls who are seated alternately in three rows, such that the first and the last positions in each row are occupied by boys. Which section could this be?

(a) *A* (b) *B* (c) *C* (d) *D*

Ans.: (b)

Solution:



If Boys & Girls are seated alternatively and first and last position is occupied by boys (possible ways)

$BGB \rightarrow 3;\quad BGBGB \rightarrow 5;\quad BGBGBGB \rightarrow 7$

This is possible when number of students in each row is odd. And, only in section B, students in each row are odd in numbers.

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Q4. A plant grows by 10% of its height every three months. If the plant's height today is 1 m, its height after one year is the closest to

(a) 1.10 m (b) 1.21 m (c) 1.33 m (d) 1.46 m

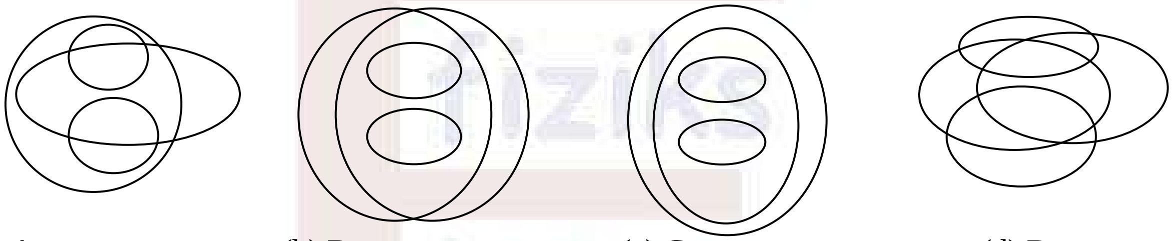
Ans.: (d)

Solution: Let 3 months = one time span; $\therefore$ 12 months = 4 time span

$$\therefore \text{ Growth after 4 time span } = 1\times\left(1+\frac{10}{100}\right)^4 = (1\cdot 1)^4 = 1.4641\text{m} \approx 1.46$$

Q5. The correct pictorial representation of the relations among the categories PLAYERS, FEMALE CRICKETERS, MALE FOOTBALLERS and GRADUATES is

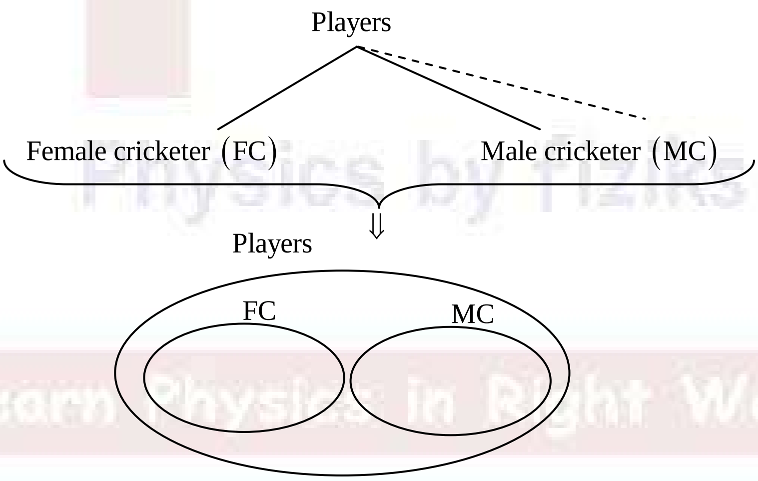
A B C D



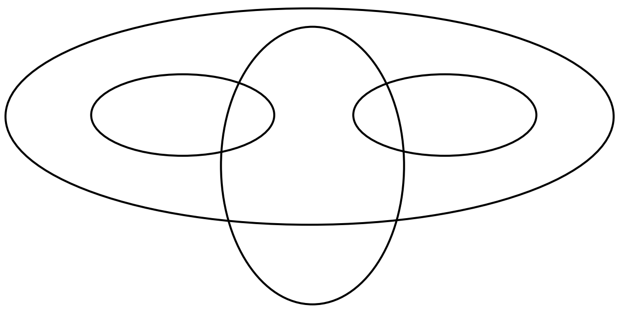
(a) A (b) B (c) C (d) D

Ans.: (a)

Solution:



Also, some players may be graduate & some not so, right choice is (a).

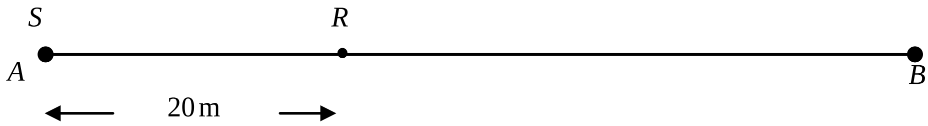


Q6. On a track of 200 m length, *S* runs from the starting point and *R* starts 20 m ahead of *S* at the same time. Both reach the end of the track at the same time. *S* runs at a uniform speed of 10 m/s. If *R* also runs at a uniform speed, what is *R*'s speed (in m/s)?

(a) 9 (b) 10 (c) 12 (d) 8

Ans.: (d)

Solution:



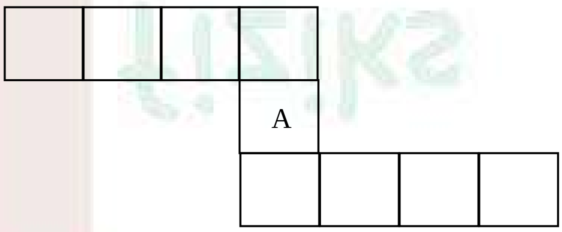
$S = v \times t$; Time = const.

$\therefore$ Ratio of speeds of S and R is proportional to distance covered by them

$v_S : v_R = 100 : 80 = 5 : 4$ $\quad\therefore\ v_R = 8\,\text{m/s}$

Distance covered by: $S = 100\,\text{m}$; $R = 80\,\text{m}$

Q7. The squares in the following sketch are filled with digits 1 to 9, without any repetition, such that the numbers in the two horizontal rows add up to 20 each. What number appears in the square labelled A in the vertical column?



(a) It cannot be ascertained in the absence of the sum of the numbers in the column

(b) 3

(c) 5

(d) 7

Ans.: (c)

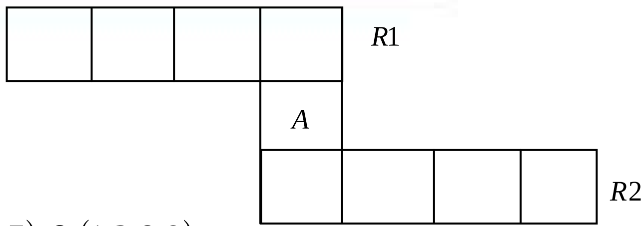
Solution:

Sum of numbers in row $R_1 = 20$ and row $R_2 = 20$

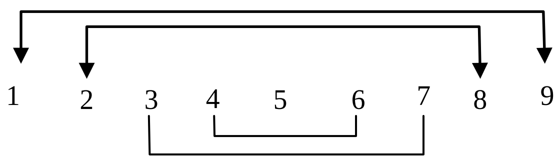
$\therefore$ Sum of nos. in R_1 & $R_2 = 40$

Also, sum of all nos. from 1 to 9 = 45

$\therefore\ A = 45 - 40 = 5$



Possible combination of nos. is Rows: $(3,4,6,7)$ & $(1,2,8,9)$.



Q8. Tokens numbered from 1 to 25 are mixed and one token is drawn randomly. What is the probability that the number on the token drawn is divisible either by 4 or by 6?

(a) 8/25 (b) 10/25 (c) 9/25 (d) 12/25

Ans.: (a)

Solution:

$n(4)$ = tokens with nos. divisible by 4 = 6.

$n(6)$ = tokens with nos. divisible by 6 = 4

$n(4\&6)$ = tokens with nos. divisible by 4 & 6 = 2 (12, 24)

$\therefore\ n(4\,\text{or}\,6) = n(4) + n(6) - n(4\,\text{and}\,6) = 6 + 4 - 2 = 8$

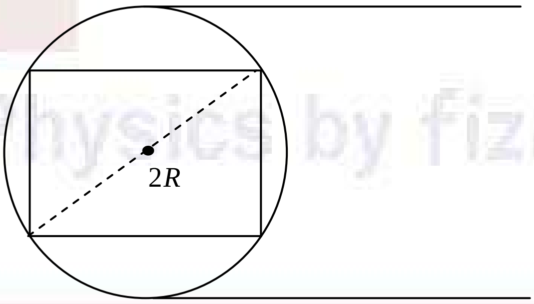
$\therefore$ Required probability $= \dfrac{8}{25}$

Q9. A beam of square cross-section is to be cut out of a wooden log. Assuming that the log is cylindrical, what approximately is the largest fraction of the wood by volume that can be fruitfully utilized as the beam?

(a) 49% (b) 64% (c) 71% (d) 81%

Ans.: (b)

Solution:



Let the radius of cylindrical log = R and length $= \ell$

$\therefore$ Largest cross-section with square as shape will have diagonal = Diameter of cylinder,

$\therefore$ Let side of square $= a$; $\therefore$ Diagonal of square $= \sqrt{2}a$

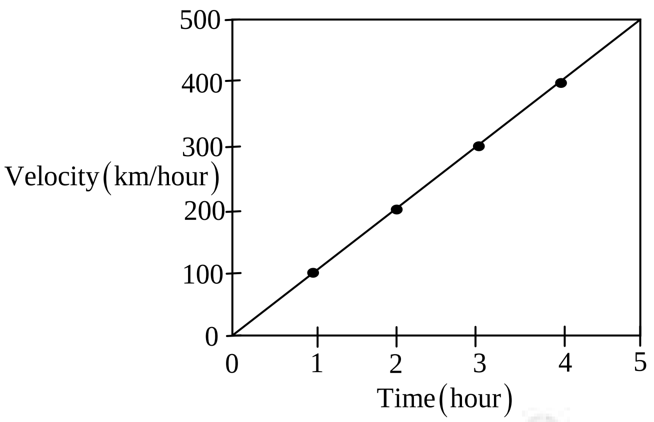
$\sqrt{2}a = 2R \Rightarrow a = \sqrt{2}R$

Fruitful largest volume $= a^2 \times \ell = \left(\sqrt{2}R\right)^2 \times \ell = 2R^2\ell$

Volume of cylinder $= \pi R^2 \ell$; $\therefore$ Fraction $= \dfrac{2R^2\ell}{\pi R^2 \ell} = \dfrac{2}{\pi}$

$$y = \frac{2}{\pi} \times 100 = \frac{200}{\pi} \approx 64\%$$

Q10. Given plot describes the motion of an object with time.



(a) The object is moving with a constant velocity.

(b) The object covers equal distance every hour.

(c) The object is accelerating.

(d) Velocity of the object doubles every hour.

Ans.: (c)

Solution:

Choice (a) : Not possible, object has different velocity at different time.

Choice (b): Distance covered in 1st hour

Area under curve $=\frac{1}{2}\times 1\times 100=50\ \text{km}$

Distance covered in first 2 hour $=\frac{1}{2}\times 2\times 200=200\ \text{km}$

Distance covered between 1st and 2nd hour $=200-50=150\ \text{km}$

$\therefore$ Not correct choice.

Choice (d) is also incorrect.

$\therefore$ Right choice is (c).

Q11. In a four-digit PIN, the third digit is the product of the first two digits and the fourth digit is zero. The number of such PINs is

(a) 42 (b) 41

(c) 40 (d) 39

Ans.: (a)

Solution:

$\underline{a}$ $\underline{b}$ $\underline{a \times b}$ $\underline{0}$

For, $a \times b = 0$

Total nos. of pairs of $(a,b) = 19$

$\underline{a}$	$\underline{b}$	$\underline{a \times b}$	
0	0	0	= 1
1	0	0	
2	0	0	
3	0	0	
4	0	0	
5	0	0	Total = 9
6	0	0	
7	0	0	
8	0	0	
9	0	0	

Being PIN, (a,b) & (b,a) are different.

$\therefore$ So, total no. of such PIN with $a \times b = 0$ is 19.

For $a \times b = 1$ Total PIN = 1 $(1,1)$;

$a \times b = 2$ Total PIN = 2 $(1,2) \& (2,1)$;

$a \times b = 3$ Total PIN = 2 $(1,3) \& (3,1)$;

$a \times b = 4$ Total PIN = 3 $(1,4),(4,1),(2,2)$;

$a \times b = 5$ Total PIN = 2 $(1,5);(5,1)$

$a \times b = 6$ Total PIN = 4 $(a,6),(6,1),(2,3),(3,2)$;

$a \times b = 7$ Total PIN = 2

$a \times b = 8$ Total PIN = 4 $(1,8),(8,1),(2,4),(4,2)$;

$a \times b = 9$ Total PIN = 3 $(1,9),(9,1),(3,3)$

$\therefore$ Total $= 19 + 23 = 42$.

Q12. I have a brother who is 4 years elder to me, and a sister who was 5 years old when my brother was born. When my sister was born, my father was 24 years old. My mother was 27 years old when I was born. How old (in years) were my father and mother, respectively, when my brother was born?

(a) 29 and 23 (b) 27 and 25

(c) 27 and 23 (d) 29 and 25

Ans.: (a)

Solution:

Let age of "Me" $=x$, Elder brother $=x+4$, Sister $=x+4+5$, Father $=x+4+5+24$

Mother $=x+27$

$\therefore$ When brother was born:

Father's age $=(x+4+5+24)-(x+4)=29$

Mother's age $=(x+27)-(x+9)=23$.

Q13. A liar always lies and a non-liar, never. If in a group of n persons seated around a round-table everyone calls his/her left neighbor a liar, then

(a) all are liars.

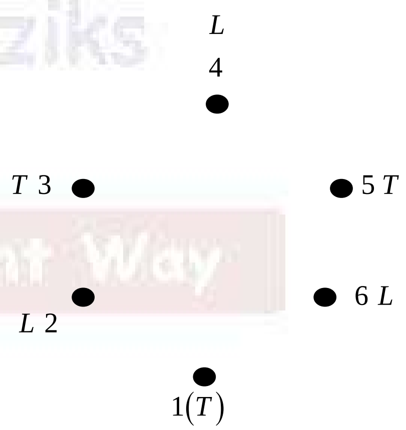
(b) n must be even and every alternate person is a liar

(c) n must be odd and every alternate person is a liar

(d) n must be a prime

Ans. : (b)

Solution:



Suppose (1) is non-liar (T), $\therefore 2 \rightarrow$ Liar (L)

2 tells 3 is liar, and 2 himself is liar.

This implies 3 is T.

3 tells 4 is liar, and 3 is T, this implies, 4 is liar and so on.

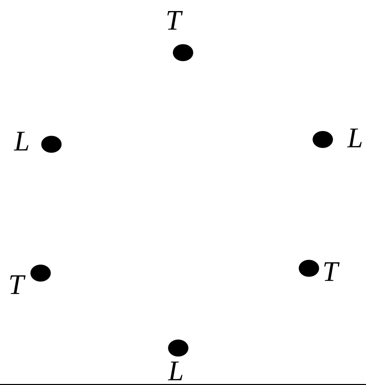
This is possible when their no. is even and every alternate is a liar.

Suppose (1) is Liar (L)

Then possible Diagram (Like above):

Again we see, the same thing.

$\therefore$ Correct choice (b).



Q14. A boy has kites of which all but 9 are red, all but 9 are yellow, all but 9 are green, and all but 9 are blue. How many kites does he have?

(a) 12 (b) 15

(c) 9 (d) 18

Ans.: (a)

Solution:

Let no. of kites $=x$, Red kites $=x-9$, Yellow kites $=x-9$, Green kite $=x-9$,

Blue kites $=x-9$

$\therefore$ Total kites: $(x-0)+(x-9)+(x-9)+(x-9)=4(x-9)$

Also, Total kites $=x$

$\therefore\ 4(x-9)=x$ or $3x=36 \Rightarrow x=12$.

Q15. If one letter each is drawn at random from the words CAUSE and EFFECT, the chance that they are the same is

(a) 1/30 (b) 1/11

(c) 1/10 (d) 2/11

Ans.: (c)

Solution:

$\underline{C}\,AU\,S\,\underline{E}$; $\underline{E}\,F\,F\,\underline{E}\,\underline{C}\,T$

Total Letters = 5 + 6 = 11

Common letters: 'C' & 'E'

Probability of pickup 'C' from CAUSE $=\frac{1}{5}$

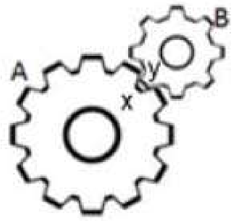
Probability of picking C from EFFECT $=\frac{1}{6}$

$\therefore$ Probability of picking 'C'' from both $=\frac{1}{5}\cdot\frac{1}{6}=\frac{1}{30}$

Similarly, probability of picking 'E' from CAUSE & EFFECT is $=\frac{1}{5}\cdot\frac{2}{6}=\frac{2}{30}$

$\therefore$ Probability of picking 'C' or 'E' $=\frac{1}{30}+\frac{2}{30}=\frac{3}{30}=\frac{1}{10}$

Q16. A vehicle has tyres of diameter 1 m connected by a shaft directly to gearwheel A which meshes with gearwheel B as shown in the diagram. A has 12 teeth and B has 8. If points x on A and y on B are initially in contact, they will again be in contact after the vehicle has travelled a distance (in meters)



(a) 2π (b) 3π (c) 4π (d) 12π

Ans.: (a)

Solution:

Point x & y will be together when A has made 2 rounds & B three rounds.

$\therefore$ Distance covered in two rounds by A: $= 2\cdot(2\pi\cdot r) = 2\cdot\left(2\pi\cdot\frac{1}{2}\right) = 2\pi$.

Q17. After 12:00:00 the hour hand and minute hand of a clock will be perpendicular to each other for the first time at

(a) 12:16:21 (b) 12:15:00

(c) 13:22:21 (d) 12:48:08

Ans.: (a)

Solution:

Let at $12:x$, minute & hour hand be at 90°.

Angle made by: Hour hand in x min $= \frac{x^0}{2}$; Minute hand in $x = 6x^0$

$$\therefore \left|6x - \frac{x}{2}\right| = 90° \Rightarrow \frac{11x}{2} = 90° \qquad \therefore x = \frac{180}{11} = 16\frac{4}{11} = 16:21$$

$\therefore$ They will be perpendicular at $12:16:21$.

Q18. What is the product of the number of capital letters and the number of small letters of the English alphabet in the following text?

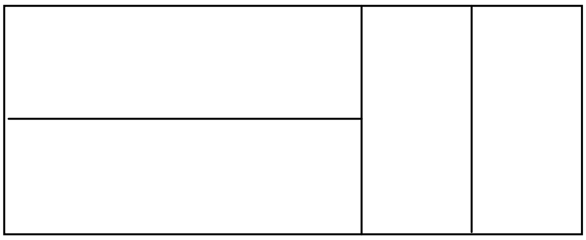
A4;={c8%$56((+B/;,.H&r]]](u];#~K@>83<??/STvx%^(d)L:/<-N347)))2;:$+}E$###[w}`”..;/89

(a) 17 (b) 37

(c) 53 (d) 63

Ans.: (d)

Q19. How many rectangles are there in the given figure?



(a) 6 (b) 7

(c) 8 (d) 9

Ans.: (c)

Q20. In a round-robin tournament, after each team has played exactly four matches, the number of wins / losses of 6 participating teams are as follows

Team	Win	Loss
A	4	0
B	0	4
C	3	1
D	2	2
E	0	4
F	3	1

Which of the two teams have certainly NOT played with each other?

(a) A and B (b) C and F

(c) E and D (d) B and E

Ans.: (d)

Solution:

B and E have containing not played with each other, had they been, at least one would have, But both are win-less.

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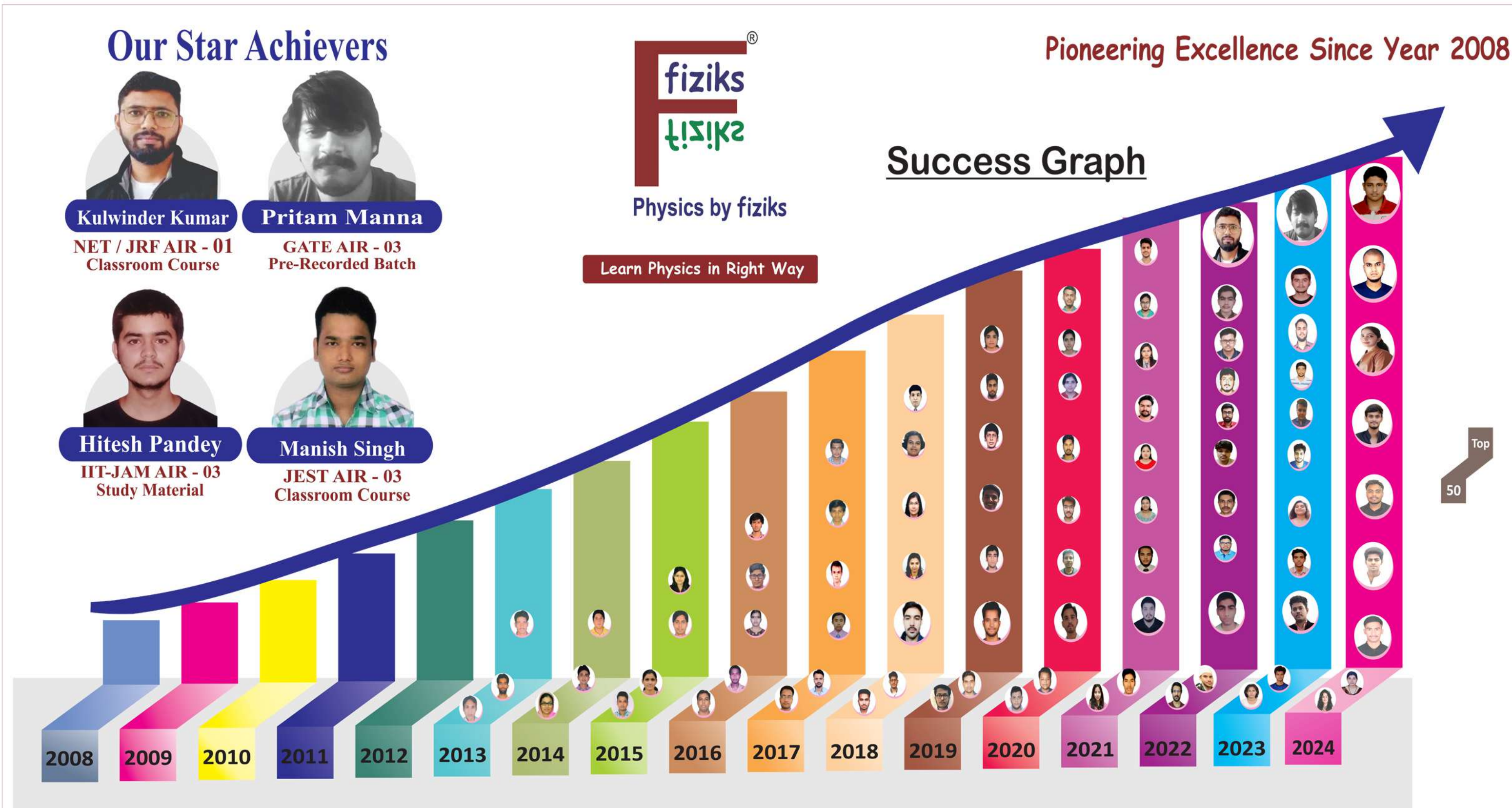

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