TIFR-2019 (Mathematical Physics Question and Solution)

SECTION A-(For both Int. Ph.D. and Ph.D. candidates)

Q1. Consider the surface defined by $ax^2+by^2+cz+d=0$, where a,b,c and d are constants. If $\hat{n}_1$ and $\hat{n}_2$ are unit normal vectors to the surface at the points $(x,y,z)=(1,1,0)$ and $(0,0,1)$ respectively and $\hat{m}$ is a unit vector normal to both $\hat{n}_1$ and $\hat{n}_2$, then $\hat{m}=$

(a) $\dfrac{-a\hat{i}+b\hat{j}}{\sqrt{a^2+b^2}}$ (b) $\dfrac{b\hat{i}-a\hat{j}}{\sqrt{a^2+b^2}}$ (c) $\dfrac{2a\hat{i}+2b\hat{j}-c\hat{k}}{\sqrt{4a^2+4b^2+c^2}}$ (d) $\dfrac{a\hat{i}+b\hat{j}-c\hat{k}}{\sqrt{a^2+b^2+c^2}}$

Ans. : (b)

Solution:

The given surface is $\phi=ax^2+by^2+cz+d=0$

$$\vec{\nabla}\phi=\left(\hat{x}\frac{\partial}{\partial x}+\hat{y}\frac{\partial}{\partial y}+\hat{z}\frac{\partial}{\partial z}\right)\left(ax^2+by^2+cz+d\right)\Rightarrow\vec{\nabla}\phi=2ax\,\hat{x}+2by\,\hat{y}+c\,\hat{z}$$

Now $\vec{\nabla}\phi$ is normal to a constant ϕ surface.

$$\vec{\nabla}\phi\Big|_{1,1,0}=2a\hat{x}+2b\hat{y}+c\hat{z},\quad \vec{\nabla}\phi\Big|_{0,0,1}=0\hat{x}+0\hat{y}+c\hat{z}=c\hat{z}$$

$$\hat{n}_1=\frac{2a\hat{x}+2b\hat{y}+c\hat{z}}{\sqrt{4a^2+4b^2+c^2}},\qquad \hat{n}_2=\frac{c\hat{z}}{\sqrt{c^2}}=\hat{z}$$

Now $\hat{n}$ is a unit vector normal to $\hat{n}_1$ and $\hat{n}_2$

Now the dot product of $\hat{n}_2=\hat{z}$ with (c) and (d) is not zero so they are ruled out.

The dot product of $\hat{n}_2=\hat{z}$ with (a) and (b) is zero. So, both are possible

So, let's check the dot product of $\hat{n}_1$ with (a) and (b)

With (a)

$$\hat{n}_1\cdot\frac{-a\hat{x}+b\hat{y}}{\sqrt{a^2+b^2}}=\frac{2a\hat{x}+2b\hat{y}+c\hat{z}}{\sqrt{4a^2+4b^2+c^2}}\cdot\frac{-a\hat{x}+b\hat{y}}{\sqrt{a^2+b^2}}=\frac{-2a^2+2b^2}{\sqrt{4a^2+4b^2+c^2}\,\sqrt{a^2+b^2}}\neq 0 \text{ in general}$$

With (b)

$$\hat{n}_1\cdot\frac{b\hat{x}-a\hat{y}}{\sqrt{a^2+b^2}}=\frac{2a\hat{x}+2b\hat{y}+c\hat{z}}{\sqrt{4a^2+4b^2+c^2}}\cdot\frac{b\hat{x}-a\hat{y}}{\sqrt{a^2+b^2}}=\frac{2ab-2ab}{\sqrt{4a^2+4b^2+c^2}\,\sqrt{a^2+b^2}}=0$$

Hence (b) is the correct answer.

Q2. The eigenvalues of a 3×3 matrix M are

$$\lambda_1=2\quad \lambda_2=-1\quad \lambda_3=1$$

and the eigenvectors are

$$e_1=\begin{pmatrix}1\\1\\1\end{pmatrix}\quad e_2=\begin{pmatrix}1\\1\\-2\end{pmatrix}\quad e_3=\begin{pmatrix}1\\-1\\0\end{pmatrix}$$

The matrix M is

(a) $\begin{pmatrix}1&0&1\\0&1&1\\1&1&0\end{pmatrix}$ (b) $\begin{pmatrix}0&1&1\\1&0&0\\1&0&2\end{pmatrix}$ (c) $\begin{pmatrix}1&0&0\\1&0&-1\\0&-1&1\end{pmatrix}$ (d) $\begin{pmatrix}1&1&0\\1&0&1\\0&1&1\end{pmatrix}$

Ans. : (a)

Solution:

Product of eigenvalues $= 2\times(-1)\times 1 = -2$

Now Det of the matrix = Product of Eigen values

$$Det(a)=\begin{vmatrix}1&0&1\\0&1&1\\1&1&0\end{vmatrix}=1\begin{bmatrix}1&1\\1&0\end{bmatrix}+1\begin{bmatrix}0&1\\1&1\end{bmatrix}=-1+(-1)=-2 \text{ Possible}$$

$$Det(b)=\begin{vmatrix}0&1&1\\1&0&0\\1&0&2\end{vmatrix}=-1\begin{bmatrix}1&0\\1&2\end{bmatrix}+1\begin{bmatrix}1&0\\1&0\end{bmatrix}=-1\times2+0=-2 \text{ Possible}$$

$$Det(c)=\begin{vmatrix}1&0&0\\1&0&-1\\0&-1&1\end{vmatrix}=1\begin{bmatrix}0&-1\\-1&1\end{bmatrix}=0-(-1\times(-1))=-1 \text{ Ruled out}$$

$$Det(d)=\begin{vmatrix}1&1&0\\1&0&1\\0&1&1\end{vmatrix}=1\begin{bmatrix}0&1\\1&1\end{bmatrix}-1\begin{bmatrix}1&1\\0&1\end{bmatrix}=-1-1=-2 \text{ Possible}$$

Thus (a), (b), (d) are possible

Now we will have to test eigenvalue equation for each of these

(a) $$\begin{pmatrix}1&0&1\\0&1&1\\1&1&0\end{pmatrix}\begin{pmatrix}1\\1\\1\end{pmatrix}=\begin{pmatrix}1\\1\\1\end{pmatrix}=\begin{pmatrix}1+0+1\\0+1+1\\1+1+0\end{pmatrix}=2\begin{pmatrix}1\\1\\1\end{pmatrix}-\text{OK}$$

$$\begin{pmatrix}1&0&1\\0&1&1\\1&1&0\end{pmatrix}\begin{pmatrix}1\\1\\-2\end{pmatrix}=\begin{pmatrix}1+0-2\\0+1-2\\1+1\end{pmatrix}=\begin{pmatrix}-1\\-1\\2\end{pmatrix}=-1\begin{pmatrix}1\\1\\-2\end{pmatrix}-\text{OK}$$

$$\begin{pmatrix}1&0&1\\0&1&1\\1&1&0\end{pmatrix}\begin{pmatrix}1\\-1\\0\end{pmatrix}=\begin{pmatrix}1+0+0\\0-1+0\\1-1+0\end{pmatrix}=1\begin{pmatrix}1\\-1\\0\end{pmatrix}-\text{OK}$$

(a) satisfies eigen value equation for all its eigen vectors.

(b) $\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1+1+0 \\ 1+0+0 \\ 1+0+2 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} \neq 2\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ Not reproduced

Hence 'b' is ruled out

(c) $\begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1+1+0 \\ 1+0+1 \\ 0+1+1 \end{pmatrix} = 2\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ – OK

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 1+1 \\ 1+0-2 \\ 1-2 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix} \neq -1\begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$$

Hence 'c' is also ruled out

Thus (a) is the correct answer.

Q3. A set of polynomials of order n are given by the formula

$$p_n(x) = (-1)^n \exp\left(\frac{x^2}{2}\right)\frac{d^n}{dx^n}\exp\left(-\frac{x^2}{2}\right)$$

The polynomial $p_7(x)$ of order $n = 7$ is

(a) $x^7 - 21x^5 + 105x^4 + 35x^3 - 105x$

(b) $x^6 - 21x^5 + 105x^4 - 105x^3 + 21x^2 + x$

(c) $x^7 - 21x^5 + 105x^3 - 105x + 21$

(d) $x^7 - 21x^5 + 105x^3 - 105x$

Ans. : (d)

Solution:

We need to find $P_7(x)$

Explicit solving can be cumbersome.

So, let's try some trick. For that too, we need to find at least P_0, P_1, P_2, P_3.

(1) $P_0(x) = (-1)^0 e^{\frac{x^3}{2}} \frac{d^0}{dx^0}\left(e^{-\frac{x^3}{2}}\right) = 1\, e^{\frac{x^2}{2}-\frac{x^2}{2}} = e^0 = 1$

Degree 0, coeff. of x^0 is 1, independent of x

(2) $P_1(x) = (-1)^1 e^{\frac{x^2}{2}} \frac{d}{dx}\left(e^{-\frac{x^2}{2}}\right) = -1x\, e^{\frac{x^2}{2}}\left[e^{-\frac{x^2}{2}} x - \frac{\cancel{2}x}{\cancel{2}}\right]$ (First derivative in square brackets)

$= x$

$\rightarrow$ Degree 1, coeff. of highest power of x is 1, ends in single power of x.

$$P_2(x)=(-1)^2 e^{\frac{x^2}{2}}\frac{d^2}{dx^2}\left(e^{-\frac{x^2}{2}}\right)=e^{\frac{x^2}{2}}\frac{d}{dx}\left[\frac{d}{dx}e^{-\frac{x^2}{2}}\right]=e^{\frac{x^2}{2}}\frac{d}{dx}\left[-xe^{-x^2/2}\right]$$

$$=+e^{\frac{x^2}{2}}\left[-x\times e^{-\frac{x^2}{2}}\times\left(\frac{-\cancel{2}x}{\cancel{2}}\right)-1e^{\frac{-x^2}{2}}\right]$$

$$=e^{\frac{x^2}{2}}\left[x^2e^{\frac{-x^2}{2}}-e^{-\frac{x^2}{2}}\right] \text{ [Second Derivative in square brackets]}$$

$$=x^2-1,$$

Degree 2, coeff. of highest power of x is 1, ends in constant and powers of x decrease by 2.

$$P_3(x)=(-1)^3 e^{\frac{x^2}{2}}\frac{d}{dx}\left[\frac{d^2}{dx^2}\left(e^{-\frac{x^2}{2}}\right)\right]\quad=-1e^{\frac{x^2}{2}}\frac{d}{dx}\left[x^2e^{-\frac{x^2}{2}}-e^{-\frac{x^2}{2}}\right]$$

$$=(-1)e^{\frac{x^2}{2}}\left[2xe^{-\frac{x^2}{2}}+x^2e^{-\frac{x^2}{2}}\times\left(\frac{-\cancel{2}x}{\cancel{2}}\right)-e^{-\frac{x^2}{2}}\times\left(\frac{-\cancel{2}x}{\cancel{2}}\right)\right]$$

$$=-1e^{\frac{x^2}{2}}\left[2xe^{-\frac{x^2}{2}}-x^2e^{-x^2/2}+xe^{-\frac{x^2}{2}}\right]$$

$$=-1e^{\frac{x^2}{2}}\left[-x^3e^{-x^2/2}+3xe^{-x^2/2}\right] \text{ [Third Derivative in square brackets]}$$

$$P_3(x)=x^3-3x,$$

Degree 3, coeff. of highest power of x is 1, ends in first power of x and powers of x decrease by 2.

Thus, in P_7, we expect,

(1) Degree to be 7

(2) Coefficient of x^7 to be 1

(3) Powers of x to decrease by 2

(4) Last term should end in single power of x

(5) Adjacent terms should be positive and negative

Let's analyze the given solutions

(a). $x^7-21x^5+105x^4+35x^3-105x$, Degree 7, coefficient of $x^7=1$, ends in single power of x, but powers of x do not decrease by 2. Hence ruled out.

(b). $x^6-21x^5+105x^4-105x^3+21x^2+x$

Degree 6, powers do not decrease by 2, though coefficient of $x^6 = 1$ and ends in x^1. Hence ruled out.

(c). $x^7 - 21x^5 + 105x^3 - 105x + 21$

Degree 7, coefficient of $x^7 = 1$, sign alternates but does not end in x^1. Power of x do not decrease by 2. Hence ruled out.

(d). $x^7 - 21x^5 + 105x^3 - 105x$

Degree 7, coefficient of $x^7 = 1$, sign alternatives

Power of x decreases by 2 and ends in x^1. Satisfies all requirements.

Thus 'd' is expected to be the correct answer.

SECTION B- (only for Int.-Ph.D. candidates)

Q4. The differential equation $x\dfrac{dy}{dx} - xy = \exp(x)$, where $y = e^2$ at $x = 1$, has the solution $y =$

(a) $\exp(x^2 + x)$ (b) $(1-x)\exp(x) + \exp(1+x)$

(c) $\exp(1+x)(1+\ln x)$ (d) $\exp(x)\ln x + \exp(1+x)$

Ans. : (d)

Solution:

$\dfrac{dy}{dx} - y = \dfrac{e^x}{x}$; Compare with $\dfrac{dy}{dx} + Py = Q$

$P = -1,\ I.F. = e^{\int Pdx} = e^{\int -dx} = e^{-x}$

Multiplying both sides by e^{-x} we get

$$e^{-x}\frac{dy}{dx} - e^{-x}y = \frac{e^{-x+x}}{x} = \frac{1}{x} \quad \Rightarrow \frac{d}{dx}\left(ye^{-x}\right) = \frac{1}{x}$$

Integrating both sides we get; $ye^{-x}\ln x + C$

Put $y = e^2$ at $x = 1$, we get $e^2 e^{-1} = \ln 1 + C \Rightarrow C = e$

Thus $ye^{-x} = \ln x + e \ \Rightarrow y = \dfrac{\ln x + e}{e^{-x}} = e^x \ln x + e^{x+1}$

Which is same as (d)

Hence (d) is the correct answer

SECTION B-(Only for Ph.D. candidates)

Q5. The integral

$$I = \int_0^{\infty} dx\, e^{-x} \delta(\sin x)$$

where $\delta(x)$ denotes the Dirac delta function, is

(a) 1 (b) $\frac{\exp\pi}{\exp\pi+1}$ (c) $\frac{\exp\pi}{\exp\pi-1}$ (d) $\frac{1}{\exp\pi-1}$

Ans. : (c)

Solution:

Normally, $\delta(\sin x) = \sum_{n=-\infty}^{\infty} \delta(x - n\pi)$

However, x only varies from 0 to ∞ in the integral to be evaluated.

$$\Rightarrow \delta(\sin x) = \sum_{n=0}^{\infty} \delta(x - n\pi)$$

Thus $I = \int_0^{\infty} dx\, e^{-x} \sum_{n=0}^{\infty} \delta(x - n\pi)$

$$I = \int_0^{\infty} dx\, e^{-x} \left[\delta(x-0\pi) + \delta(x-1\pi) + \delta(x-2\pi)\right) + \ldots\ldots\ldots\ldots + \delta(x-n\pi)$$

$$I = \int_0^{\infty} dx\, e^{-x} \delta(x-0\pi) + \int_0^{\infty} dx\, e^{-x}\, \delta(x-\pi) + \int_0^{\infty} dx\, e^{-x} \delta(x-2\pi) \ldots\ldots + \int_0^{\infty} dx\, e^{-x} \delta(x-n\pi)$$

Using $\int_{-\infty}^{\infty} \delta(x-a)\phi(x)\, dx = \phi(a)$

$a \to 0,\ \ \pi - n\pi\ ;\ \ \phi(x) = e^{-x}$

$I = e^0 + e^{-\pi} + e^{-2\pi} + \ldots.. + e^{-n\pi}$

This is an infinite G.P. with first term as 1 and common ratio as $e^{-\pi}$. Thus,

$$I = \frac{a}{1-M},\ a = 1,\ M = e^{-\pi} \qquad \Rightarrow I = \frac{1}{1-e^{-\pi}} = \frac{e^{\pi}}{e^{\pi}-1}$$

Hence (c) is the answer.

H.No. 40-D, G.F., Jia Sarai, Near IIT, Hauz Khas, New Delhi-16
Phone: 011-26865455/+91-9871145498, Website: www.physicsbyfiziks.com, Email: fiziks.physics@gmail.com

Q6. Consider the complex function

$$f(x,y)=u(x,y)+iv(x,y)$$

where $$u(x,y)=x^2(2+x)-y^2(2+3x)$$

$$v(x,y)=y\left(\lambda x+3x^2-y^2\right)$$

and λ is real. If it is known that $f(x,y)$ is analytic in the complex plane of $z=iy$, then it can be written

(a) $f=z^2(2+z)$ (b) $f=\bar{z}\left(2+\bar{z}^2\right)$

(c) $f=2z\bar{z}+z^2-\bar{z}^2$ (d) $f=z^2+z^3$

Ans. : (a)

Solution: As function is analytic so $\dfrac{\partial u}{\partial x}=\dfrac{\partial v}{\partial y}$

$$4x+\cancel{3x^2}-\cancel{3y^2}=\lambda x+\cancel{3x^2}-\cancel{3y^2}\Rightarrow\lambda=4$$

So $f(x,y)=\left(2x^2+x^3-2y^2-3xy^2\right)+i\left[4xy+3x^2y-y^3\right]$

Let's test the options

(a) $f=z^2(2+z)=(x+iy)^2(2+x+iy)=2(x+iy)^2+(x+iy)^3$ $\{x+iy=z\}$

$$=2\left(x^2-y^2+2ixy\right)+x^3-iy^3+i3x^2y-3xy^2$$

Collect real and imaginary parts.

$$f=\left(2x^2+x^3-2y^2-3xy^2\right)+i\left[4xy+3x^2y-y^3\right]$$

Thus (a) has exactly reproduced (1)

We can similarly show that (b), (c) and (d) do not reproduce (1).

The answer is hence (a).

TIFR-2019 (EMT Question and Solution)

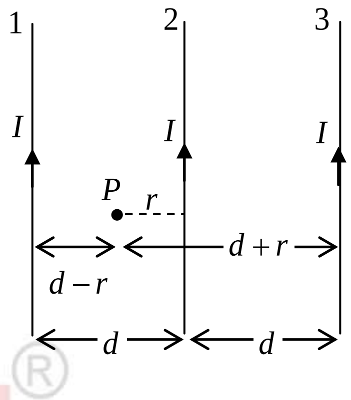
SECTION A-(For both Int. Ph.D. and Ph.D. candidates)

Q1. Consider three straight coplanar, parallel wires of infinite length where the distance between adjacent where is d. Each wire carries a current I in the same direction. The perpendicular distance from the middle wire (on either side) where the magnetic field vanishes is

(a) $\dfrac{d}{\sqrt{3}}$ (b) $\dfrac{2d}{\sqrt{3}}$ (c) $\dfrac{d}{3}$ (d) $\dfrac{2d}{3}$

Ans: (a)

Solution:



Magnetic field due to wire 1 at point P is

$$B_1 = \frac{\mu_0 I}{2\pi(d-r)} \text{ (pointing inward)}$$

Magnetic field due to wire 2 at point P is

$$B_2 = \frac{\mu_0 I}{2\pi r} \text{ (pointing outward)}$$

Magnetic field due to wire 3 at point P is

$$B_3 = \frac{\mu_0 I}{2\pi(d+r)} \text{(pointing outward)}$$

Magnetic field due to wires at point P will vanish if

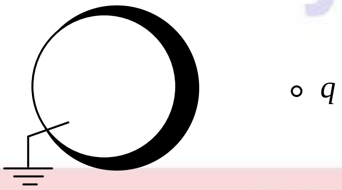
$$B_1 = B_2 + B_3 \Rightarrow \frac{\mu_0 I}{2\pi(d-r)} = \frac{\mu_0 I}{2\pi r} + \frac{\mu_0 I}{2\pi(d+r)}$$

$$\Rightarrow \frac{1}{(d-r)} = \frac{1}{r} + \frac{1}{(d+r)} \Rightarrow \frac{1}{d-r} = \frac{d+2r}{r(d+r)} \Rightarrow rd + r^2 = d^2 + 2rd - rd - 2r^2$$

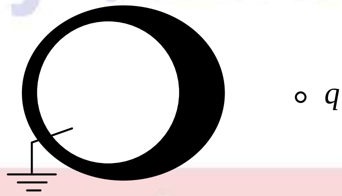
$$\Rightarrow r^2 = d^2 - 2r^2 \Rightarrow 3r^2 = d^2 \Rightarrow r = \frac{d}{\sqrt{3}}$$

Q2. A point charge $q < 0$ is brought in front of a grounded conducting sphere. If the induced charge density on the sphere is plotted such that the thickness of the black shading is proportional to the charge, density the correct plot will most closely resemble

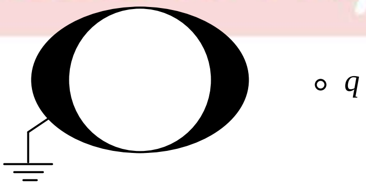
(a)



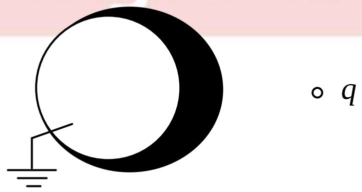
(b)



(c)



(d)

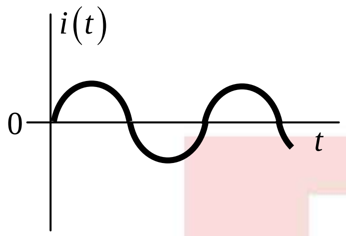


Ans: (b)

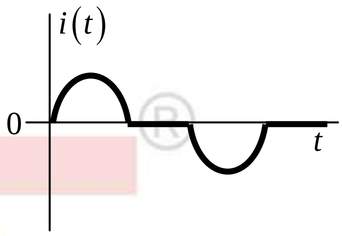
Solution: There is always a net attraction between a charge and conductor, so charge on conductor will be more towards charge q.

Q3. A circular coil of conducting wire of radius a and n turns, is placed in a uniform magnetic field $\vec{B}$ along the axis of the coil and is then made to undergo simple harmonic oscillations along the direction of the axis. The current through the coil will be best described by

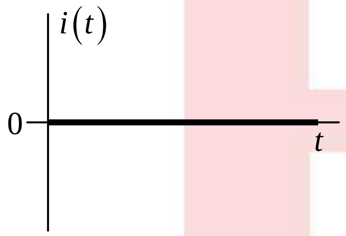
(a)



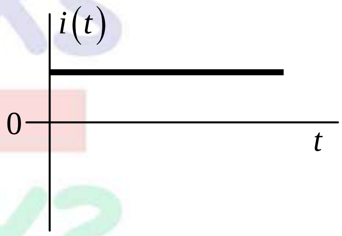
(b)



(c)



(d)



Ans: (c)

Solution: The coil is made to undergo simple harmonic oscillations along the direction of the axis in uniform magnetic field. So magnetic flux is constant. Thus current is zero.

Q4. A plane electromagnetic wave travelling through vacuum has electric field $\vec{E}$ and magnetic field $\vec{B}$ defined as

$$\vec{E} = \left(\hat{i} + \hat{j}\right) E_0 \exp i\left(\omega t - \vec{k} \cdot \vec{x}\right) \qquad \vec{B} = \left(\hat{i} - \hat{j} - \hat{k}\right) B_0 \exp i\left(\omega t - \vec{k} \cdot \vec{x}\right)$$

where E_0 and B_0 are real constants. The time-averaged pointing vector will be given by

(a) $\vec{S} = -\frac{2}{\sqrt{\epsilon_0 \mu_0}} E_0 B_0 \left(\hat{i} - \hat{j} + 2\hat{k}\right)$

(b) $\vec{S} = -\frac{1}{2}\sqrt{\frac{3\epsilon_0}{\mu_0}} E_0^2 \left(\hat{i} - \hat{j} + 2\hat{k}\right)$

(c) $\vec{S} = \sqrt{\frac{\epsilon_0}{6\mu_0}} E_0^2 \left(-\hat{i} + \hat{j} - 2\hat{k}\right)$

(d) $\vec{S} = \frac{1}{2}\sqrt{\frac{\epsilon_0}{\mu_0}} B_0^2 \left(\hat{i} - \hat{j} + 2\hat{k}\right)$

Ans. : (c)

Solution:

Let us write real part of electric and magnetic field.

$$\vec{E} = \left(\hat{i} + \hat{j}\right) E_0 \exp i\left(\omega t - \vec{k} \cdot \vec{x}\right) = \left(\hat{i} + \hat{j}\right) E_0 \cos\theta, \qquad \text{Let } \theta = \left(\omega t - \vec{k} \cdot \vec{x}\right).$$

$$\vec{B} = \left(\hat{i} - \hat{j} - \hat{k}\right) B_0 \exp i\left(\omega t - \vec{k} \cdot \vec{x}\right) = \left(\hat{i} - \hat{j} - \hat{k}\right) B_0 \cos\theta$$

Poynting Vector $\vec{S}=\frac{1}{\mu_0}\left(\vec{E}\times\vec{B}\right)=\frac{1}{\mu_0}\left[\left(\hat{i}+\hat{j}\right)E_0\cos\theta\right]\times\left[\left(\hat{i}-\hat{j}-\hat{k}\right)B_0\cos\theta\right]$

$$\Rightarrow \vec{S}=\frac{1}{\mu_0}\left[\left(\hat{i}+\hat{j}\right)\times\left(\hat{i}-\hat{j}-\hat{k}\right)\right]\times\left[E_0B_0\cos^2\theta\right]=\frac{1}{\mu_0}\left(-\hat{i}+\hat{j}-2\hat{k}\right)\times E_0\left(\frac{E_0}{c}\right)\cos^2\theta$$

$\because B_0=\frac{E_0}{c}$

Time average of Poynting Vector is

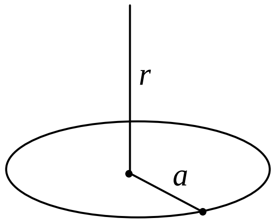
$$\left\langle\vec{S}\right\rangle=\frac{1}{\mu_0}\left(-\hat{i}+\hat{j}-2\hat{k}\right)\times E_0^2\left(\sqrt{\varepsilon_0\mu_0}\right)\times\frac{1}{2}\qquad \because\left\langle\cos^2\theta\right\rangle=\frac{1}{2}\text{ and } c=\frac{1}{\sqrt{\varepsilon_0\mu_0}}$$

$$\left\langle\vec{S}\right\rangle=\sqrt{\frac{\varepsilon_0}{4\mu_0}}E_0^2\left(-\hat{i}+\hat{j}-2\hat{k}\right)$$

So best option is (c).

SECTION B- (only for Int.-Ph.D. candidates)

Q5. Consider a hydrogenic atom in its ground state ass conceived in Bohr's theory, where an electron of charge $-e$ is rotating about a central nucleus of charge $+Ze$ in a circular orbit of radius $a=4\pi\in_0\hbar^2/Ze^2$. In this model, the magnetic field at a distance r from the nucleus, perpendicular to the orbit will be



(a) $\frac{Ze^3\mu_0}{16\pi^2\hbar\in_0}\frac{1}{a}\left(1+\frac{r^2}{a^2}\right)^{-1/2}$

(b) $\frac{Ze^2\mu_0}{4\pi^2\hbar\in_0}\frac{1}{a}\left(1+\frac{r^2}{a^2}\right)^{-1/2}$

(c) $\frac{Ze\mu_0}{\hbar\in_0}\left(1+\frac{r^2}{a^2}\right)^{-3/2}$

(d) $\frac{Ze^3\mu_0}{8\pi^2\hbar\in_0}\frac{1}{a}\left(1+\frac{r^2}{a^2}\right)^{-3/2}$

Ans. :

Solution. :

The magnetic field at a distance r from the nucleus, perpendicular to the orbit is

$$B=\frac{\mu_0 I}{2}\frac{a^2}{\left(a^2+r^2\right)^{3/2}}\Rightarrow B=\frac{\mu_0 I}{2}\frac{1}{a\left(1+\frac{r^2}{a^2}\right)^{3/2}}\quad\text{where } I=\frac{e}{T}=\frac{e}{2\pi a/v}=\frac{ev}{2\pi a}.$$

$$\because mv_nr_n=n\hbar\Rightarrow mva=\hbar\quad\Rightarrow v=\frac{\hbar}{ma}\quad\because n=1$$

Thus $B=\frac{\mu_0}{2}\left(\frac{ev}{2\pi a}\right)\frac{1}{a}\left(1+\frac{r^2}{a^2}\right)^{-3/2}=\frac{\mu_0}{2}\left(\frac{e}{2\pi a}\times\frac{\hbar}{ma}\right)\frac{1}{a}\left(1+\frac{r^2}{a^2}\right)^{-3/2}$

$$\because a=4\pi\in_0\hbar^2/Ze^2m\quad\Rightarrow\frac{\hbar}{ma}=\frac{Ze^2}{4\pi\in_0\hbar}$$

Thus $B=\frac{\mu_0}{2}\left(\frac{e}{2\pi a}\times\frac{Ze^2}{4\pi\in_0\hbar}\right)\frac{1}{a}\left(1+\frac{r^2}{a^2}\right)^{-3/2}\Rightarrow B=\frac{Ze^3\mu_0}{16\pi^2\hbar a^2\in_0}\left(1+\frac{r^2}{a^2}\right)^{-3/2}$

No answer exactly matches.

Q6. A dielectric interface is formed by two homogeneous and isotropic dielectrics 1 and 2 with dielectric constants $\frac{4}{3}$ and 1 respectively and it carries no residual free charge. A linearly polarized electromagnetic wave is incident on the interface from dielectric1 at a point where the unit normal to the surface is

$$\hat{n}=\frac{1}{2}\left(\sqrt{3}\hat{i}+\hat{k}\right)$$

Pointing into the dielectric1. The incident wave, which is incident from 1 into 2, just before it reaches the interface, has electric vector

$$\vec{E}_1=\hat{i}E_0\exp i\omega\left(t+\frac{\sqrt{\varepsilon_1}}{c}z\right)$$

where E_0 is a real constant. The electric vector just after it crosses the interface is rotated from $\vec{E}_1$ by an angle

(a) $\frac{\pi}{6}$ (b) $\tan^{-1}\frac{2}{5\sqrt{3}}$ (c) $\sin^{-1}\frac{1}{2\sqrt{19}}$ (d) $\csc^{-1}3\sqrt{\frac{4}{19}}$

Ans.: (c)

Solution. :

$$\vec{E}_1=\hat{i}E_0\exp i\omega\left(t+\frac{\sqrt{\varepsilon_1}}{c}z\right)\text{ and }\hat{n}=\frac{1}{2}\left(\sqrt{3}\hat{i}+\hat{k}\right).$$

Normal component of electric field in medium 1 is

$$\left|\vec{E}_{1n}\right|=\left(\vec{E}_1\right).\hat{n}=\frac{\sqrt{3}}{2}E_0\exp i\omega\left(t+\frac{\sqrt{\varepsilon_1}}{c}z\right)$$

Thus $\vec{E}_{1n}=\frac{\sqrt{3}}{2}E_0\exp i\omega\left(t+\frac{\sqrt{\varepsilon_1}}{c}z\right)\hat{n}=\frac{\sqrt{3}}{2}E_0\exp i\omega\left(t+\frac{\sqrt{\varepsilon_1}}{c}z\right)\frac{1}{2}\left(\sqrt{3}\hat{i}+\hat{k}\right)$

Thus normal component of electric field in medium 2 is

$$\vec{E}_{2n}=\frac{\varepsilon_1\vec{E}_{1n}}{\varepsilon_2}=\frac{4/3\varepsilon_0}{1\varepsilon_0}\vec{E}_{1n}=\frac{4}{3}\times\frac{\sqrt{3}}{4}E_0\exp i\omega\left(t+\frac{\sqrt{\varepsilon_1}}{c}z\right)\left(\sqrt{3}\hat{i}+\hat{k}\right)=E_0\exp i\omega\left(t+\frac{\sqrt{\varepsilon_1}}{c}z\right)\frac{1}{\sqrt{3}}\left(\sqrt{3}\hat{i}+\hat{k}\right)$$

Tangential component of electric field in medium 1 is

$$\vec{E}_{1t}=\vec{E}_1-\vec{E}_{1n}=E_0\exp i\omega\left(t+\frac{\sqrt{\varepsilon_1}}{c}z\right)\left[\hat{i}-\frac{\sqrt{3}}{2}\frac{1}{2}\left(\sqrt{3}\hat{i}+\hat{k}\right)\right]=E_0\exp i\omega\left(t+\frac{\sqrt{\varepsilon_1}}{c}z\right)\left[\frac{1}{4}\hat{i}-\frac{\sqrt{3}}{4}\hat{k}\right]$$

Thus tangential component of electric field in medium 2 is

$$\vec{E}_{2t} = \vec{E}_{1t} = E_0 \exp i\omega\left(t + \frac{\sqrt{\varepsilon_1}}{c}z\right)\left[\frac{1}{4}\hat{i} - \frac{\sqrt{3}}{4}\hat{k}\right]$$

The electric field in medium 2 is

$$\vec{E}_2 = \vec{E}_{2t} + \vec{E}_{2n} = E_0 \exp i\omega\left(t + \frac{\sqrt{\varepsilon_1}}{c}z\right)\left[\left(\frac{1}{4}\hat{i} - \frac{\sqrt{3}}{4}\hat{k}\right) + \frac{1}{\sqrt{3}}\left(\sqrt{3}\hat{i} + \hat{k}\right)\right]$$

$$\Rightarrow \vec{E}_2 = E_0 \exp i\omega\left(t + \frac{\sqrt{\varepsilon_1}}{c}z\right)\left(\frac{5}{4}\hat{i} + \frac{1}{4\sqrt{3}}\hat{k}\right)$$

Let angle between $\vec{E}_1$ and $\vec{E}_2$ be θ then; $\vec{E}_1.\vec{E}_2 = \left|\vec{E}_1\right|\left|\vec{E}_2\right|\cos\theta$

$$\Rightarrow \left[E_0 \exp i\omega\left(t + \frac{\sqrt{\varepsilon_1}}{c}z\right)\right]^2 \times \frac{5}{4} = \left[E_0 \exp i\omega\left(t + \frac{\sqrt{\varepsilon_1}}{c}z\right)\right]^2 \sqrt{\frac{25}{16} + \frac{1}{48}}\cos\theta$$

$$\Rightarrow \frac{5}{4} = \frac{1}{4}\sqrt{25 + \frac{1}{3}}\cos\theta \Rightarrow 5 = \sqrt{\frac{76}{3}}\cos\theta \Rightarrow \cos\theta = \frac{5\sqrt{3}}{\sqrt{76}}$$

$$\Rightarrow \sin\theta = \sqrt{1 - \cos^2\theta} = \sqrt{1 - \frac{75}{76}} = \sqrt{\frac{1}{76}} = \frac{1}{2\sqrt{19}}$$

$$\Rightarrow \theta = \sin^{-1}\left(\frac{1}{2\sqrt{19}}\right)$$

SECTION B-(Only for Ph.D. candidates)

Q7. The magnetic vector potential corresponding to a uniform magnetic field $\vec{B}$ is often taken as

$$\vec{A} = \frac{1}{2}\vec{B} \times \vec{x}$$

This choice is

(a) valid in the Lorenz gauge

(b) valid in the Coulomb gauge

(c) valid in the Weyl gauge

(d) gauge invariant

Ans.: (b)

Solution:

Let $\vec{B} = B_1\hat{x} + B_2\hat{y} + B_3\hat{z}$ and $\vec{x} = x\hat{x}$

$$\vec{A} = \frac{1}{2}\vec{B} \times \vec{x} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ B_1 & B_2 & B_3 \\ x & 0 & 0 \end{vmatrix} = \frac{1}{2}\left[\hat{x}(0-0) - \hat{y}(0 - B_3x) + \hat{z}(0 - B_2x)\right]$$

$$\Rightarrow \vec{A} = \frac{1}{2}\left[B_3x\hat{y} - B_2x\hat{z}\right]$$

Thus $\vec{\nabla}.\vec{A} = 0$.

TIFR-2019 (Quantum Mechanics Question and Solution)

SECTION A-(For both Int. Ph.D. and Ph.D. candidates)

Q1. A particle of mass m, moving in one dimension, satisfies the modified Schrödinger equation

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2}+i\hbar u\frac{d\psi}{dx}=i\hbar\frac{d\psi}{dt}$$

Where u is the velocity of the substrate? If, now, this particle is treated as a Gaussian wave packet peaked at wave number k, its group velocity will be $v_g =$

(a) $\frac{hk}{2m}-u$ (b) $\frac{hk}{m}+u$ (c) $\frac{hk}{m}-u$ (d) $-\frac{hk}{m}+u$

Ans. :(c)

Solution:

A particle of mass m, moving in one dimension, satisfies the modified Schrödinger Equation

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2}+i\hbar u\frac{d\psi}{dx}=i\hbar\frac{d\psi}{dt} \qquad (1)$$

Rearranging the above equation, we get

$$-\frac{1}{2m}\left(-\hbar^2\frac{d^2}{dx^2}\right)\psi+(-u)\left(-i\hbar\frac{d}{dx}\right)\psi=\left(i\hbar\frac{d}{dt}\right)\psi$$

$$\frac{1}{2m}p^2\psi+(-u)p=E\psi\Rightarrow\frac{p^2}{2m}-pu=E \qquad (2)$$

Where, we have used

$$p=-i\hbar\frac{\partial}{\partial x}\ ;\ \ p^2=-\hbar^2\frac{\partial^2}{\partial x^2}\ \ ;\ \ E=i\hbar\frac{\partial}{\partial t}. \qquad (3)$$

Expressing eq. (2) in terms of angular frequency and momentum, we get

$$E=\frac{p^2}{2m}-pu\Rightarrow\hbar\omega=\frac{\hbar^2k^2}{2m}-\hbar ku \quad \text{or} \quad \omega=\frac{\hbar k^2}{2m}-ku$$

The group velocity of the particle is defined as,

$$v_g=\frac{d\omega}{dk}=\frac{d}{dk}\left(\frac{\hbar k^2}{2m}-ku\right)=\frac{\hbar k}{m}-u$$

Q2. In a one-dimensional system, the boundary condition that the derivative of the wavefunction $\psi'(x)$ should be continuous at every point is applicable whenever

(a) The wavefunction $\psi(x)$ is itself continuous everywhere.

(b) There is a bound state and the potential is piecewise continuous.

(c) There is a bounded state and the potential has no singularity anywhere.

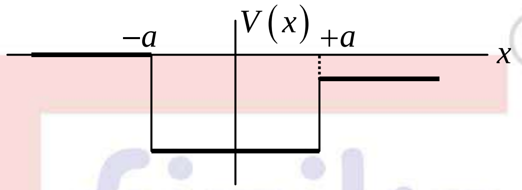
(d) There are bound or scattering states with definite momentum.

Ans. :(c)

Solution:

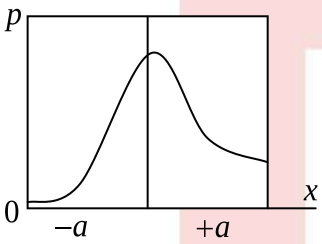
In a one-dimensional system, the boundary condition that the derivative of the wave function $\psi'(x)$ should be continuous at every point is applicable whenever there is a bound state and the potential has no singularity anywhere.

Q3. A particle moving in one dimension, is placed in an asymmetric square well potential $V(x)$ as sketched below.

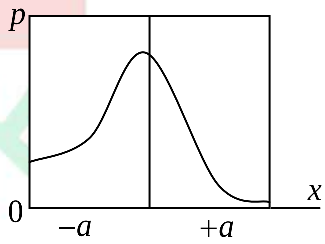


The probability density $p(x)$ in the ground state will most closely resemble

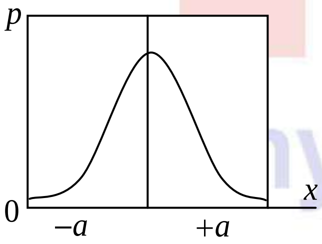
(a)



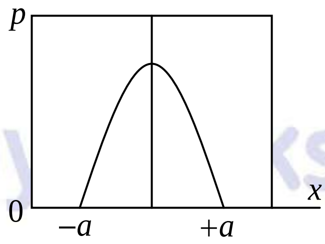
(b)



(c)



(d)



Ans. :(a)

Solution:

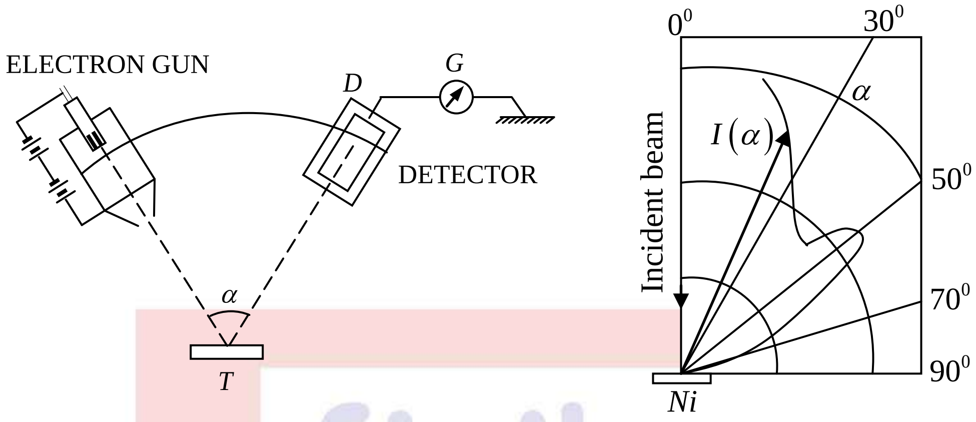
This problem is solved through options and using elimination methods.

(a) Since the potential is not symmetrical, Hence the probability density can not be symmetrical which means option (b) and (d) are incorrect.

(b) Option (c) is eliminated as the particle can not move in the region $x > -a$ due to the height of the potential.

(c) Thus, we are left with option (a), which is correct. Further, the particle has finite chances to move in the region $x > a$. Hence there is probability of finding the particle in the region $x > a$. This is correctly represented in the option (a).

Q4. The sketch shown below illustrates the apparatus and results for a famous experiment. The graph on the right is a polar plot of the number of electrons received in the detector.



(a) The energy levels of atoms in a metal are quantized.

(b) Electrons in a beam can behave as waves.

(c) Electrons have spin half.

(d) There are magnetic domains inside a nickel sample.

Ans. :(b)

Solution:

This present apparatus given in the question is for the davisson germer experiment which proved the wave nature of the electrons.

SECTION B- (only for Int.-Ph.D. candidates)

Q5. An excited gas, consisting of spinless charged particles, is confined in an infinite square well potential of width a, is found to radiate a spectrum whose α line (largest wavelength) has wavelength $816\ nm$. If the width a of the well is halved to $\frac{a}{2}$, the wavelength of the δ line (fourth-largest wavelength) will be

(a) $26.112\ \mu m$ (b) $1.224\ \mu m$ (c) $1.088\ g\mu m$ (d) $0.306\ \mu m$

Ans. : none of the given option is correct.

Solution:

The energy of the spin less charged particle in an excited gas confined in an infinite square well potential of width $'a'$ is given by

$$E = n^2 \frac{\pi^2\hbar^2}{2ma^2} \qquad n = 1, 2, 3, ... \qquad (1)$$

The wave length of α line emitted by the particle is given by

$$\frac{hc}{\lambda_\alpha} = E_u - E_l = \left(2^2 - 1^2\right)\frac{\pi^2\hbar^2}{2ma^2} = \frac{3\pi^2\hbar^2}{2ma^2} \qquad (2)$$

The wavelength of δ line emitted by the particle when width a is reduced to $a/2$ is given by

$$\frac{hc}{\lambda_\beta}=E_u-E_l=\left(5^2-4^2\right)\frac{\pi^2\hbar^2}{2m\left(a/2\right)^2}=36\frac{\pi^2\hbar^2}{2ma^2} \qquad (3)$$

On dividing eq.(2) by eq(3), we get

$$\frac{\left(hc/\lambda_\alpha\right)}{\left(hc/\lambda_\beta\right)}=\frac{\left(\pi^2\hbar^2/2ma^2\right)}{\left(\pi^2\hbar^2/2ma^2\right)}\frac{3}{36} \text{ or } \frac{\lambda_\beta}{\lambda_\alpha}=\frac{3}{36}\Rightarrow\lambda_\beta=\frac{\lambda_\alpha}{12}=\frac{816}{12}=68\,nm$$

Q6. The wave function of a non-relativistic particle of mass m in a one-dimensional potential $V(x)$ has the form

$$\psi(x)=\sqrt{a}e^{-a|x|}$$

Where $|x|$ denotes the absolute value of the coordinate x. The potential is $V(x)=$

(a) $-\frac{\hbar^2}{m}\delta\left(\frac{x}{a}\right)$

(b) $-\frac{\hbar^2 a}{m}\left\{a+\frac{1}{2}\delta(x)\right\}$

(c) $\frac{\hbar^2 a}{m}\delta(x)$

(d) 0

Ans. :(a)

Solution:

For dirac delta potential having from $V(x)=-\delta\left(\frac{x}{a}\right)=-a\delta(x)$.

The wave function of the particle is of the from $\psi(x)=\frac{\sqrt{ma}}{\hbar}e^{-\frac{ma}{\hbar^2}|x|}$

Thus, for the potential $V(x)=-\frac{\hbar^2}{m}\delta\left(\frac{x}{a}\right)=-\frac{\hbar^2}{m}a\,\delta(x)$.

The wave function of the particle is given by

$$\psi(x)=\pm\sqrt{a}\,e^{-a|x|}$$

Note: $|x|$ term in the exponential can only occur if the attractive potential of the particle is in the from of direct delta function

H.No. 40-D, G.F., Jia Sarai, Near IIT, Hauz Khas, New Delhi-16
Phone: 011-26865455/+91-9871145498, Website: www.physicsbyfiziks.com, Email: fiziks.physics@gmail.com

SECTION B-(Only for Ph.D. candidates)

Q7. An electron in a hydrogen atom is in a state described by the wavefunction:

$$\psi(\vec{r})=\frac{1}{\sqrt{10}}\psi_{100}(\vec{x})+\sqrt{\frac{2}{5}}\psi_{210}(\vec{x})+\sqrt{\frac{2}{5}}\psi_{211}(\vec{x})-\frac{1}{\sqrt{10}}\psi_{21,-1}(\vec{x})$$

where $\psi_{nlm}(\vec{x})$ denotes a normalized wavefunction of the hydrogen atom with the principal quantum number n, angular quantum number ℓ and magnetic quantum number m. Neglecting the spin-orbit intersection, the expectation values of $\hat{L}_z$ and $\hat{L}^2$ for this state are

(a) $\frac{3\hbar}{10},\frac{9\hbar^2}{5}$ (b) $\frac{3\hbar}{4},\frac{9\hbar^2}{25}$ (c) $\frac{3\hbar}{5},\frac{9\hbar^2}{10}$ (d) $\frac{8\hbar}{10},\frac{3\hbar^2}{5}$

Ans. :(a)

Solution:

An electron in a hydrogen atom is in a state described by the wave function:

$$|\psi(r)\rangle=\frac{1}{\sqrt{10}}|1,0,0\rangle+\sqrt{\frac{2}{5}}|2,1,0\rangle+\sqrt{\frac{2}{5}}|2,1,1\rangle-\frac{1}{\sqrt{10}}|2,1,-1\rangle$$

The wave function $|\psi(r)\rangle$ is normalized as

$$\langle\psi(r)|\psi(r)\rangle=\frac{1}{10}+\frac{2}{5}+\frac{2}{5}+\frac{1}{10}=1$$

The expedition value of L^2 is determined as follows.

$$L^2|\psi(r)=\frac{1}{\sqrt{10}}L^2|1,0,0\rangle+\sqrt{\frac{2}{5}}L^2|2,1,0\rangle+\sqrt{\frac{2}{5}}L^2|2,1,1\rangle-\frac{1}{\sqrt{10}}L^2|2,1,-1\rangle$$

$$=\frac{1}{\sqrt{10}}\hbar^2 0.(0+1)|1,0,0\rangle+\sqrt{\frac{2}{5}}\hbar^2 1(1+1)|2,1,0\rangle+\sqrt{\frac{2}{5}}\hbar^2 1.(1+1)|2,1,1\rangle-\frac{1}{\sqrt{10}}\hbar^2 1(1+1)|2,1,-1\rangle$$

$$L^2|\psi(r)\rangle=(2\hbar^2)\left[\sqrt{\frac{2}{5}}|2,1,0\rangle+\sqrt{\frac{2}{5}}|2,1,1\rangle-\frac{1}{\sqrt{10}}|2,1,-1\rangle\right]$$

or

$$\langle\psi(r)|L^2\psi(r)\rangle=(2\hbar^2)\left[\frac{2}{5}\langle 2,1,0|2,1,0\rangle+\frac{2}{5}\langle 2,1,1|2,1,1\rangle+\frac{1}{10}\langle 2,1,-1|2,1,-1\rangle\right]$$

$$\Rightarrow\langle\psi(r)|L^2\psi(r)\rangle=2\hbar^2\left[\frac{2}{5}+\frac{2}{5}+\frac{1}{10}\right]=\frac{9}{6}\hbar^2$$

Similarly, the expectation value of $\langle L_z\rangle$ is determined as follows.

$$L_z|\psi(r)\rangle=\sqrt{\frac{1}{10}}L_z|1,0,0\rangle+\sqrt{\frac{2}{5}}L_z|2,1,0\rangle+\sqrt{\frac{2}{5}}L_z|2,1,1\rangle-\frac{1}{\sqrt{10}}L_z|2,1,-1\rangle$$

$$=\sqrt{\frac{1}{10}}(0\hbar)|100\rangle+\sqrt{\frac{2}{5}}(0.\hbar)|2,1,0\rangle+\sqrt{\frac{2}{5}}(\hbar)|2,1,1\rangle-\frac{1}{\sqrt{10}}(-\hbar)|2,1,-1\rangle$$

$$=\hbar\left(\sqrt{\frac{2}{5}}|2,1,1\rangle+\sqrt{\frac{1}{10}}|2,1,-1\rangle\right)$$

or $\langle\hat{L}_z\rangle=\langle\psi(r)|L_z|\psi(r)\rangle=\hbar\left[\frac{2}{5}\langle 2,1,1|2,1,1\rangle-\frac{1}{10}\langle 2,1,-1|2,1,-1\rangle\right]$

$$\Rightarrow\langle\hat{L}_z\rangle=\hbar\left[\frac{2}{5}-\frac{1}{10}\right]=\frac{3\hbar}{10}$$

The expectation value of $\hat{L}_z$ and $\hat{L}^2$ for this states are

$$\left(L_z,L^2\right)=\frac{3\hbar}{10},\frac{9\hbar^2}{5}$$

Q8. A system of two spin - $1/2$ particles 1 and 2 has the Hamiltonian

$$H=\epsilon_0\,\hat{\hbar}_1\otimes\hat{\hbar}_2$$

where

$$\hat{\hbar}_1=\begin{pmatrix}2&0\\0&1\end{pmatrix}\quad\hat{\hbar}_2=\begin{pmatrix}0&1\\1&0\end{pmatrix}$$

and ϵ_0 is a constant with the dimension of energy. The ground state of this system has energy

(a) $\sqrt{2}\,\epsilon_0$ (b) 0 (c) $-2\,\epsilon_0$ (d) $-4\,\epsilon_0$

Ans. :(c)

Solution:

A system of two spin-$1/2$ particles 1 and 2 has the Hamiltonian

$$\hat{H}=\epsilon_0\,\hat{\hbar}_1\otimes\hat{\hbar}_2=\epsilon_0\begin{pmatrix}2&0\\0&1\end{pmatrix}\otimes\begin{pmatrix}0&1\\1&0\end{pmatrix}=\epsilon_0\begin{bmatrix}2\begin{pmatrix}0&1\\1&0\end{pmatrix}&0\begin{pmatrix}0&1\\1&0\end{pmatrix}\\0\begin{pmatrix}0&1\\1&0\end{pmatrix}&1\begin{pmatrix}0&1\\1&0\end{pmatrix}\end{bmatrix}$$

$$\hat{H}=\begin{bmatrix}0&2\epsilon_0&0&0\\2\epsilon_0&0&0&0\\0\quad 0&&0&\epsilon_0\\0\quad 0&&\epsilon_0&0\end{bmatrix}$$

Where, we have used, $\hat{h}_1=\begin{pmatrix}2&0\\0&1\end{pmatrix}$; $\hat{h}_2=\begin{pmatrix}0&1\\1&0\end{pmatrix}$

The ground state energy of the system is given by

$$|H-\lambda I| = \begin{vmatrix} -\lambda & 2\in_0 & 0 & 0 \\ 2\in_0 & -\lambda & 0 & 0 \\ 0 & 0 & -\lambda & \in_0 \\ 0 & 0 & \in_0 & -\lambda \end{vmatrix} = 0$$

$$\Rightarrow -\lambda \begin{bmatrix} -\lambda & 0 & 0 \\ 0 & -\lambda & \in_0 \\ 0 & \in_0 & -\lambda \end{bmatrix} - 2\in_0 \begin{vmatrix} 2\in & 0 & 0 \\ 0 & -\lambda & \in_0 \\ 0 & \in_0 & -\lambda \end{vmatrix} = 0$$

$$\Rightarrow \lambda^2\left(\lambda^2 - \in_0^2\right) - 4\in_0^2\left(\lambda^2 - \in_0^2\right) = 0$$

$$\left(\lambda^2 - (2\in_0)^2\right)\left(\lambda^2 - \in_0^2\right) = 0 \Rightarrow \lambda = \pm 2\in_0, \pm\in_0 \,.$$

From the given option, the ground state energy of the system is $-2\in_0$`.

Q9. At time $t=0$, the wavefunctionof a particle in a harmonic oscillator potential of natural frequency ω is given by

$$\psi(0) = \frac{1}{5}\left\{3\varphi_0 - 2\sqrt{2}\varphi_1 + 2\sqrt{2}\varphi_2\right\}$$

where $\varphi_n(x)$ denotes the eigenfunction belonging to the n-th eigenvalue of energy. At time $t=\tau$, the wavefunction is found to be

$$\psi(\tau) = \frac{i}{5}\left\{3\varphi_0 + 2\sqrt{2}\varphi_1 + 2\sqrt{2}\varphi_2\right\}$$

the minimum value of τ is

(a) $\dfrac{\pi}{2\omega}$ (b) $\dfrac{2\pi}{\omega}$ (c) $\dfrac{2\pi}{3\omega}$ (d) $\dfrac{\pi}{\omega}$

Ans. :(d)

Solution:

At time $t=0$ the wave function of a particle in a harmonic oscillator potential of natural frequency ω is given by $\psi(0) = \frac{1}{5}\left(3\psi_0 - 2\sqrt{2}\psi_1 + 2\sqrt{2}\psi_2\right)$

The given wave function is normalized as $\langle\psi(0)|\psi(0)\rangle = \frac{9}{25} + \frac{8}{25} + \frac{8}{25} = 1$

The wave function of the particle at time t is

$$\psi(t) = \frac{1}{5}\left(3\phi_0 e^{-i\frac{E_0}{\hbar}t} - 2\sqrt{2}\phi_1 e^{-i\frac{E_1}{\hbar}t} + 2\sqrt{2}\phi_2 e^{\frac{-iE}{\hbar}t}\right)$$

$$= \frac{1}{5}\left(3\phi_0 e^{i\frac{\hbar\omega}{2\hbar}t} - 2\sqrt{2}\phi_1 e^{-i\frac{3}{2}\frac{\hbar\omega}{\hbar}t} + 2\sqrt{2}\phi_2 e^{-\frac{5}{2}\frac{\hbar\omega}{\hbar}t}\right) = \frac{1}{5}\left(3\phi e^{-i\frac{\omega t}{2}} - 2\sqrt{2}\phi_1 e^{-i\frac{3}{2}\omega t} + 2\sqrt{2}\phi_2 e^{-i\frac{5}{2}\omega t}\right)$$

From the given option, we assume minimum value of τ is

$$\psi(\tau) = \frac{1}{5}\left(3\phi_0 e^{-i\frac{\omega}{2}\cdot\frac{\pi}{\omega}} - 2\sqrt{2}\phi_1 e^{-i\frac{3\omega}{2}\frac{\pi}{\omega}} + 2\sqrt{2}\phi_2 e^{-i\frac{5\omega}{2}\frac{\pi}{\omega}}\right)$$

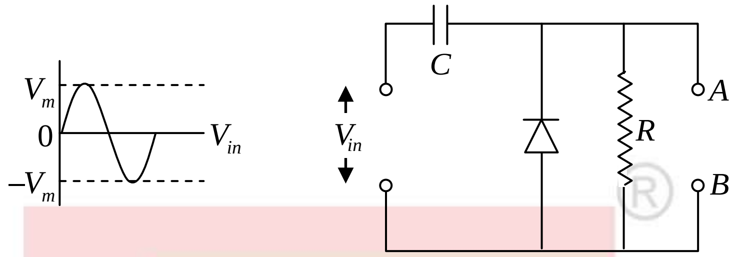
$$= \frac{1}{5}\left(3\phi_0 e^{-i\pi/2} - 2\sqrt{2}\phi_1 e^{-i3\pi/2} + 2\sqrt{2}\phi_2 e^{-i5\pi/2}\right)$$

$\psi(\tau) = -\frac{i}{5}\left(3\phi_0 - 2\sqrt{2}\phi_1 + 2\sqrt{2}\phi_2\right)$. Thus, the correct option is (d).

TIFR-2019 (Electronics Question and Solution)

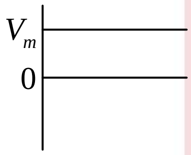
SECTION A-(For both Int. Ph.D. and Ph.D. candidates)

Q1. The signal shown on the left side of the figure below is fed into the circuit shown on the right side.

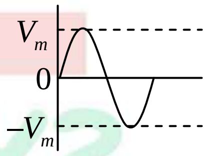


If the signal has time period τ_s and the circuit has a natural frequency τ_{RC}, then in the case when $\tau_s << \tau_{RC}$, the steady-state output will resemble

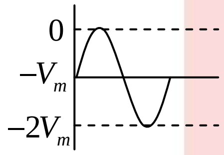
(a)



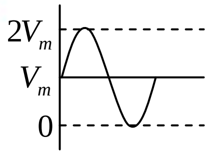
(b)



(c)



(d)

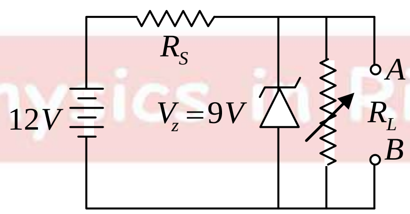


Ans: (d)

Solution:

Clamping Circuit.

Q2. Drawing power from a $12V$ car battery a $9V$ stabilized DC voltage is required to power a car stereo system, attached to the terminals A and B, as shown in the figure.



If a Zener diode with ratings $V_Z = 9V$ and $P_{\max} = 0.27W$, is connected as shown in the figure, for the above purpose, the minimum series resistance R_S must be

(a) 111Ω (b) 103Ω (c) 100Ω (d) 97Ω

Ans: (c)

H.No. 40-D, G.F., Jia Sarai, Near IIT, Hauz Khas, New Delhi-16
Phone: 011-26865455/+91-9871145498, Website: www.physicsbyfiziks.com, Email: fiziks.physics@gmail.com

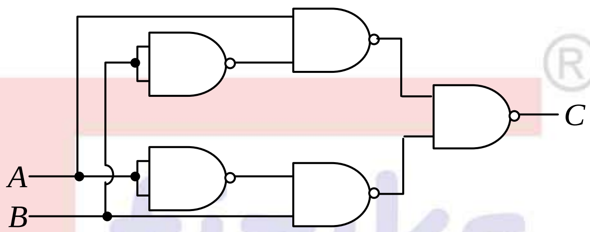
Solution:

$$P_{\max} = V_Z I_{ZM} = 9V \times I_{ZM} = 0.27W \Rightarrow I_{ZM} = 30\,mA$$

Current through R_S is $I_S = I_{ZM} + I_{L,\min} \approx I_{ZM} \Rightarrow \frac{12V - 9V}{R_S} = 30\,mA$

$$\Rightarrow R_S = 100\ \Omega$$

Q3. The circuit shown below uses only NAND gates



The final output at C is

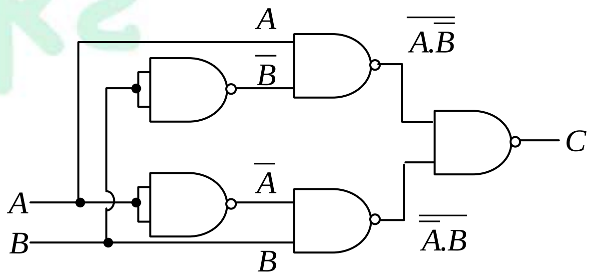
(a) A AND B (b) A OR B (c) A XOR B (d) A NOR B

Ans: (c)

Solution:

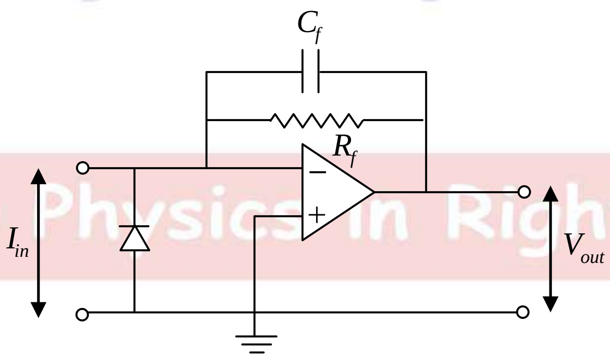
$$C = \overline{\left(\overline{A.\overline{B}}\right)\left(\overline{\overline{A}.B}\right)} = \overline{\left(\overline{A.\overline{B}}\right)} + \overline{\left(\overline{\overline{A}.B}\right)}$$

$$\Rightarrow C = A.\overline{B} + \overline{A}.B$$



SECTION B-(Only for Ph.D. candidates)

Q4. Consider the following circuit



It is given that $C_f = 100\,pF$, and for $I_{in} = 50\,nA$ $D.C$, $V_{out} = 1V\ D.C$. Therefore, the bandwidth of the above circuit is

(a) $15.8\,Hz$ (b) $79.6\,Hz$ (c) $145.3\,Hz$ (d) $200.4\,Hz$

Ans. : (b)

Solution. :

For $I_{in} = 50\,nA\ \ D.C$, $V_{\text{out}} = 1V\ D.C$.

So $\left|V_{\text{out}}\right| = R_f I_{in} \Rightarrow 1V = R_f \times \left(50\times10^{-9}A\right) \Rightarrow R_f = 20\,M\Omega$

Therefore, the bandwidth of the above circuit is

$$f = \frac{1}{2\pi R_f C_f} \Rightarrow f = \frac{1}{2\times3.14\times\left(20\times10^{6}\right)\left(100\times10^{-12}\right)} Hz \Rightarrow f = \frac{10^4}{125.6} Hz = 79.6\,Hz$$

TIFR-2019 (Solid State Physics Questions and Solution)

SECTION B-(Only for Ph.D. candidates)

Q1. At low temperature, the measured specific heat C_V of a solid sample is found to depend on temperature as $C_V = aT^{3/2} + bT^3$

Where a and b are constants. This material has

(a) One fermionic excitation with dispersion relation $\omega \propto k^4$ another bosonic excitation with dispersion relation $\omega \propto k$;

(b) One fermionic excitation with dispersion relation $\omega \propto k^2$ another bosonic excitation with dispersion relation $\omega \propto k^4$;

(c) One bosonic excitation with dispersion relation $\omega \propto k^2$ another bosonic excitation with dispersion relation $\omega \propto k$;

(d) One fermionic excitation with dispersion relation $\omega \propto k^2$ another bosonic excitation with dispersion relation $\omega \propto k$;

Ans: (c)

Solution:

For a general dispersion relation of the type

$$\omega \propto k^s$$

The specific heat in d-dimension depends on temperature as

$$C_V \propto T^{d/s}$$

The given measured specific heat C_V of a solid sample is $C_V = aT^{3/2} + bT^3 = C_1 + C_2$.

The first component $C_1 = aT^{3/2}$, gives $d = 3,\ s = 2$, the corresponding dispersion relation becomes $\omega = Ak^2$. This is a dispersion relation of magnons. A magnon is a boson and is a collective excitation of electron spin. Bosons have even spin (0,1,2,…). The first component $C_1 = aT^3$, gives $d = 3,\ s = 1$, the corresponding dispersion relation becomes $\omega = Ak$. This is a dispersion relation of phonons. A phonon is a quasi-particle and is a collective excitation of electron spin, it is a Bosons have even spin 1.

Q2. In a sample of germanium, at a temperature $77\ K$, optical excitation creates an average density of 10^{12} conduction electrons per cm^3. At this temperature, the electron and hole nobilities are equal and given by

$$\mu = 0.5\times10^4\ cm^2 s^{-1} V^{-1}$$

The value of the Einstein diffusion coefficient for the electrons and holes is

(a) $3.3\times10^{-2}\ m^2 s^{-1}$ (b) $1.65\times10^{-3}\ m^2 s^{-1}$

(c) $3.3\times10^{-3}\ m^2 s^{-1}$ (d) $6.6\times10^{-3}\ m^2 s^{-1}$

Ans: (c)

Solution:

The electron mobility is

$$\mu = 0.5\times10^4\ cm^2 s^{-1} V^{-1} = 0.5\, m^2 s^{-1} V^{-1}$$

Einstein diffusion coefficient is

$$D_n = \mu_n \frac{kT}{e} = 0.5\times\frac{1.38\times10^{-23}\times77}{1.6\times10^{-19}} = 3.3\times10^{-3} m^2 s^{-1}$$

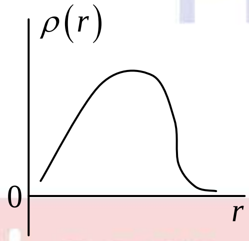
Thus, correct option is (c).

TIFR-2019 (Nuclear Physics Questions and Solution)

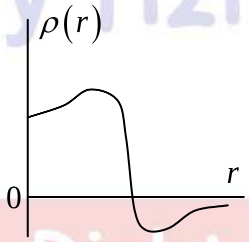
SECTION B- (only for Int.-Ph.D. candidates)

Q1. The electric charge density $\rho(r)$ inside a heavy spherical nucleus as a function of distance r from the centre may be approximated most closely by

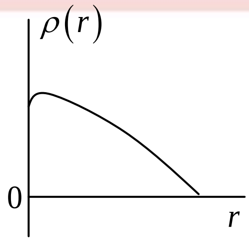
(a)



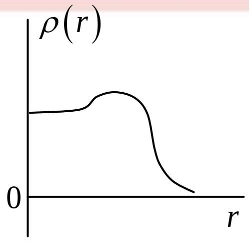
(b)



(c)



(d)



Ans.: (d)

Solution.: Based on experimental results.

SECTION B-(Only for Ph.D. candidates)

Q2. The semi-empirical mass formula for a heavy nucleon (Z, A) can be written, to some approximation, as

$$M(Z,A)c^2 = ZM_pc^2 + (A-Z)M_nc^2 - \lambda_2 A^{2/3} - \lambda_3 \frac{Z(Z-1)}{A^{1/2}} - \lambda_4 \frac{(A-2Z)^2}{A} - \frac{\lambda_5}{A^{1/2}}$$

where $M_pc^2 = 938\,MeV, M_nc^2 = 939\,MeV$, and $\lambda_1 = 16,\ \lambda_2 = 18,\ \lambda_3 = 0.7,\ \lambda_4 = 23$ all in MeV, where

$$\lambda_5 = \begin{cases} +12\,MeV \text{ for even - even nuclei} \\ -12\,MeV \text{ for odd - odd nuclei} \\ 0 \text{ for other} \end{cases}$$

now, consider a spontaneous fission reaction

$$^{238}_{92}U \rightarrow ^{146}_{56}Ba + ^{91}_{36}Kr + ^{1}_{0}n$$

the energy released in this reaction will be close to

(a) $17.92\,keV$ (b) $19.2\,MeV$ (c) $170\,MeV$ (d) $190\,MeV$

Ans.: (d)

Solution: $\Delta E = [M_U - M_{Ba} - M_{kr} - M_n]c^2$ …………..(1)

$^{238}_{92}U \rightarrow Z = 92, A - Z = 146, A - 2Z = 54$

$$M_Uc^2 = (92\times 938) + (146\times 939) - (16\times 238) - (18\times 238^{2/3}) - 0.7\frac{92\times 91}{238^{1/3}} - 23\frac{(54)^2}{238} - \frac{12}{(238)^{1/2}}$$

Similarly find the values of $M_{Ba}c^2, M_{kr}c^2$ and M_nc^2 and substitute these values into $Eq.(1)$ we get, $\Delta E = 190\,MeV$

Q3. The table below gives the properties of four unstable particles $\mu^+, \pi^+, n^0, \wedge^0$

Particle	Mass (MeV/c^2)	spin	Principal decay mode
Muon μ^+	105.66	$\frac{1}{2}$	$\mu^+ \rightarrow e^+ + v_\mu + \bar{v}_e$
Pion π^+	139.57	0	$\pi^+ \rightarrow \mu^+ + v_\mu$
Neutron n^0	939.56	$\frac{1}{2}$	$n^0 \rightarrow p^+ + e^- + \bar{v}_e$
Lambda hyperon $\wedge^0$	1,115.68	$\frac{1}{2}$	$\wedge^0 \rightarrow p^+ + \pi^-$

H.No. 40-D, G.F., Jia Sarai, Near IIT, Hauz Khas, New Delhi-16
Phone: 011-26865455/+91-9871145498, Website: www.physicsbyfiziks.com, Email: fiziks.physics@gmail.com

If arranged in order of DECREASING decay lifetime, the above list will read

(a) $n^0, \mu^+, \pi^+, \wedge^0$ (b) $\mu^+, \wedge^0, n^0, \pi^+$

(c) $n^0, \wedge^0, \mu^+, \pi^+$ (d) $\pi^+, n^0, \mu^+, \wedge^0$

Ans.: (a)

Solution:

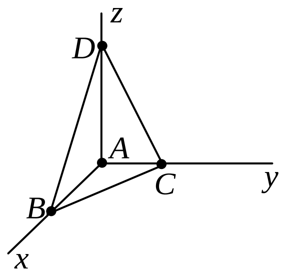
Particle	Lifetime (s)
μ^+	2.2×10^{-6}
π^+	2.2×10^{-8}
n^0	993
Λ^0	2.5×10^{-10}

General

SECTION A-(For both Int. Ph.D. and Ph.D. candidates)

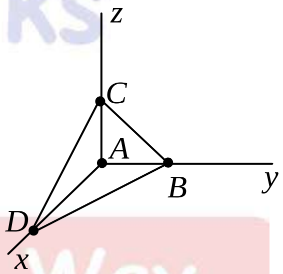
Q3. Which of the following operations will transform a tetrahedron $ABCD$ with vertices as listed below

	x	y	z
A	0	0	0
B	1	0	0
C	0	1	0
D	0	0	2



Into a tetrahedron $ABCD$ with vertices as listed below

	x	y	z
A	0	0	0
B	0	1	0
C	0	0	1
D	2	0	0



Up to suitable translation?

(a) A rotation about x axis by $\frac{\pi}{2}$ then a rotation about z axis by $\frac{\pi}{2}$

(b) A reflection in the xy plane, then a rotation about x axis by $\frac{\pi}{2}$

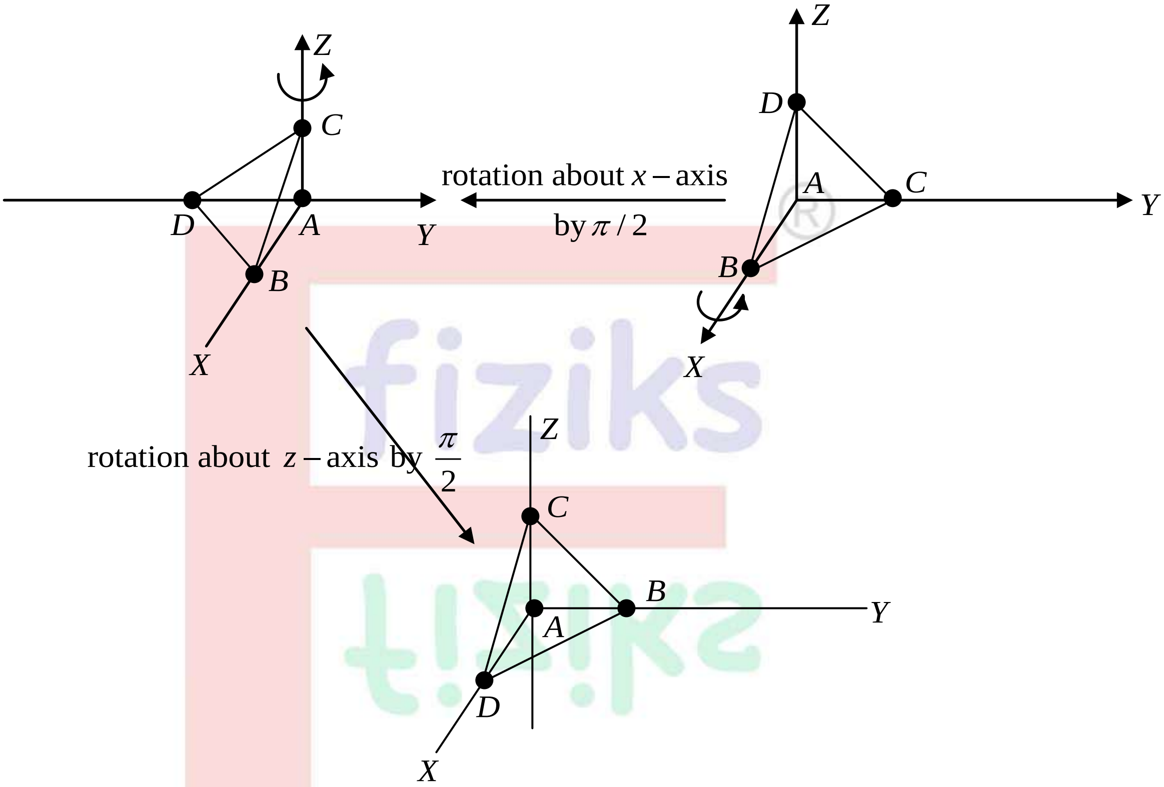
(c) A reflection in the yz plane, then a rotation about xy plane

(d) A rotation about y axis by $\frac{\pi}{2}$, then a reflection in the xz plane

Ans.: (a)

Solution:

A rotation by X axis by $\pi/_2$, then a rotation about Z-axis by $\pi/_2$.



H.No. 40-D, G.F., Jia Sarai, Near IIT, Hauz Khas, New Delhi-16
Phone: 011-26865455/+91-9871145498, Website: www.physicsbyfiziks.com, Email: fiziks.physics@gmail.com

Physics by fiziks

Pioneering Excellence Since Year 2008

Learn Physics in Right Way

Our Star Achievers

Our Star Achievers

Physics by fiziks

Learn Physics in Right Way

Pioneering Excellence Since Year 2008

011-26865455, +9871145498

Head Office: House No. 40-D, Ground Floor, Jia Sarai Near IIT-Delhi, Hauz Khas, New Delhi-110016