TIFR-2020 (Mathematical Physics Question and Solution)

SECTION A-(For both Int. Ph.D. and Ph.D. candidates)

Q1. The limit $\lim_{x\to\infty} x\log\frac{x+1}{x-1}$

Evaluate to

(a) 2 (b) 0 (c) ∞ (d) 1

Ans. : (a)

Solution:

Using the standard formula

$$\log\left(\frac{(x+1)}{(x-1)}\right)=2\left[\frac{1}{x}+\frac{1}{3x^3}+\frac{1}{5x^5}+......\right]$$

$$x\log\left(\frac{x+1}{x-1}\right)=2\left[\frac{x}{x}+\frac{x}{3x^3}+\frac{x}{5x^5}+.....\right]=2\left[1+\frac{1}{3x^2}+\frac{1}{5x^4}+.....\right]$$

$$\lim_{x\to\infty} x\log\left(\frac{x+1}{x-1}\right)=2\left[1+\frac{1}{\infty}+\frac{1}{\infty}+.....\right]=2$$

Hence the answer is (a)

Q2. The eigenvector e_1 corresponding to the smallest eigenvalue of the matrix

$$\begin{pmatrix} 2a^2 & a & 0 \\ a & 1 & a \\ 0 & a & 2a^2 \end{pmatrix}$$

where $a=\sqrt{\frac{3}{2}}$ is given (in terms of its transpose) by

(a) $e_1^T=\frac{1}{2}\left(\frac{1}{\sqrt{2}}\ -\sqrt{3}\ \frac{1}{\sqrt{2}}\right)$ (b) $e_1^T=\frac{1}{2}\left(\sqrt{\frac{3}{2}}\ \ 1\ \ \sqrt{\frac{3}{2}}\right)$

(c) $e_1^T=\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 0 & -1\end{pmatrix}$ (d) $e_1^T=\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 0 & 1\end{pmatrix}$

Ans: (a)

Solution:

To find Eigen values $A-\lambda I=0$

$$\begin{bmatrix} 2a^2 & a & 0 \\ a & 1 & a \\ 0 & a & 2a^2 \end{bmatrix}-\lambda\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}=0\Rightarrow\begin{bmatrix} 2a^2-\lambda & a & 0 \\ a & 1-\lambda & a \\ 0 & a & 2a^2-\lambda \end{bmatrix}=0$$

Opening the determinant

$$\left(2a^2-\lambda\right)\begin{bmatrix} 1-\lambda & a \\ a & 2a^2-\lambda \end{bmatrix}-a\begin{bmatrix} a & a \\ 0 & 2a^2-\lambda \end{bmatrix}=0$$

$$\left(2a^2-\lambda\right)\left[2a^2-\lambda-2a^2\lambda+\lambda^2-a^2\right]-a^2\left(2a^2-\lambda\right)=0$$

$$\left(2a^2-\lambda\right)\left[2a^2-\lambda-2a^2\lambda+\lambda^2-a^2-a^2\right]=0$$

$$\left(2a^2-\lambda\right)\left(\lambda^2-\left(2a^2+1\right)\lambda\right)=0\Rightarrow\left(2a^2-\lambda\right)\cdot\lambda\left(\lambda-\left(2a^2+1\right)\right)=0$$

$$\Rightarrow\lambda=2a^2,\ \lambda=0,\ \lambda=2a^2+1 \qquad \Rightarrow\lambda=2\cdot\frac{3}{2},\ \lambda=0,\lambda=2\cdot\frac{3}{2}+1$$

$$\Rightarrow\lambda=3,\ 0,\ 4$$

Thus, the smallest eigen value is 0.

Eigen Vector: $AX=\lambda X$ $\Rightarrow\begin{bmatrix}2\cdot\frac{3}{2} & \sqrt{\frac{3}{2}} & 0\\ \sqrt{\frac{3}{2}} & 1 & \sqrt{\frac{3}{2}}\\ 0 & \sqrt{\frac{3}{2}} & 2\cdot\frac{3}{2}\end{bmatrix}\begin{bmatrix}\alpha\\ \beta\\ \gamma\end{bmatrix}=0\begin{bmatrix}\alpha\\ \beta\\ \gamma\end{bmatrix}$

$$\begin{bmatrix}3\alpha+\sqrt{\frac{3}{2}}\beta\\ \sqrt{\frac{3}{2}}\alpha+\beta+\sqrt{\frac{3}{2}}\gamma\\ \sqrt{\frac{3}{2}}\beta+3\gamma\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix} \Rightarrow 3\alpha+\sqrt{\frac{3}{2}}\beta=0\Rightarrow\beta=\frac{-3}{\sqrt{3/2}}\alpha=-\sqrt{6}\alpha\Rightarrow\beta=-\sqrt{6}\alpha$$

$$\Rightarrow\sqrt{\frac{3}{2}}\alpha+\beta+\sqrt{\frac{3}{2}}\gamma=0\Rightarrow\sqrt{\frac{3}{2}}\beta+3\gamma=0\Rightarrow\gamma=\frac{-\sqrt{3/2}}{3}\beta=\frac{-1}{\sqrt{6}}\times\left(-\sqrt{6}\alpha\right)=\alpha\Rightarrow\gamma=\alpha$$

Thus, eigen vector is of the from $\begin{bmatrix}\alpha\\ -\sqrt{6}\alpha\\ \alpha\end{bmatrix}=\alpha\begin{bmatrix}1\\ -\sqrt{6}\\ 1\end{bmatrix}$

Normalizing, we get $\alpha^2\left[1+6+1\right]=1\Rightarrow\alpha=\frac{1}{\sqrt{8}}=\frac{1}{2\sqrt{2}}$

Thus, the eigen vector is $\begin{bmatrix}\alpha\\ \beta\\ \gamma\end{bmatrix}=\frac{1}{2\sqrt{2}}\begin{bmatrix}1\\ -\sqrt{6}\\ 1\end{bmatrix}=\frac{1}{2}\begin{bmatrix}\frac{1}{\sqrt{2}}\\ -\sqrt{3}\\ \frac{1}{\sqrt{2}}\end{bmatrix}$

Taking Transpose, Eigen vector $e_1^T=\frac{1}{2}\left(\frac{1}{\sqrt{2}}-\sqrt{3}\ \frac{1}{\sqrt{2}}\right)$

In the options this matches (a).

Q3. Consider the improper differential

$$ds = \left(1 + y^2\right) dx + xy dy$$

An integrating factor for this is

(a) $-x$ (b) $1 + x^2$ (c) xy (d) $-1 + y^2$

Ans. : (a)

Solution:

I.F. =?

Compare with $Mdx + Ndy$; $M = 1 + y^2$, $N = xy$ $\Rightarrow \frac{\partial M}{\partial y} = 2y$, $\frac{\partial N}{\partial x} = y$

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{2y - y}{xy} = \frac{y}{xy} = \frac{1}{x} = f(x) = \text{Function of 'x' alone}$$

$$I.F. = e^{\int f(x)dx} = e^{\int \frac{1}{x}dx} = e^{\ln x} = x$$

Though x is not given in options

But $-x$ is given.

Hence (a) is expected to be the right answer.

SECTION B- (only for Int.-Ph.D. candidates)

Q4. The sum of the infinite series

$$S = 1 + \frac{3}{5} + \frac{6}{25} + \frac{10}{125} + \frac{15}{625} + ...$$

is given by

(a) $S = \frac{125}{64}$ (b) $S = \frac{25}{16}$

(c) $S = \frac{25}{24}$ (d) $S = \frac{16}{25}$

Ans.: (a)

Solution. :

$$S = 1 + \frac{3}{5} + \frac{6}{25} + \frac{10}{125} + \frac{15}{625} + \cdots\cdots = 1 + \frac{3}{5} + \frac{5+1}{25} + \frac{10}{125} + \frac{15}{625} + \cdots\cdots$$

$$S = 1 + \frac{3}{5} + \frac{1}{25} + \frac{5}{25} + \frac{10}{125} + \frac{15}{625} + \cdots\cdots = 1 + \frac{3}{5} + \frac{1}{25} + \left\{\frac{1}{5} + \frac{2}{25} + \frac{3}{125} + \cdots\cdots\right\}$$

$$S = 1 + \frac{3}{5} + \frac{1}{25} + \left\{\frac{0}{5^0} + \frac{1}{5} + \frac{2}{5^2} + \frac{3}{5^3} + \cdots\cdots\right\}$$

Arthogeometric progression $S = \frac{a}{1-m} + \frac{d \cdot m}{(1-m)^2}$

$a =$ First term of A.P, $m = c.m.$ of G.P, $d = c \cdot d$ of A.P

For $a=0$; $S=\dfrac{d\cdot m}{(1-m)^2}$

$$S=1+\frac{3}{5}+\frac{1}{25}+\frac{1\cdot\frac{1}{5}}{\left(1-\frac{1}{5}\right)^2}=1+\frac{3}{5}+\frac{1}{25}+\frac{1}{5}\times\frac{25}{16}=1+\frac{3}{5}+\frac{1}{25}+\frac{1}{25}=1.925$$

SECTION B-(Only for Ph.D. candidates)

Q5. The solution of the differential equation

$$\frac{dy}{dx}=1+\frac{y}{x}-\frac{y^2}{x^2}$$

for $x>0$ with the boundary condition $y=0$ at $x=1$. is given by $y(x)=$

(a) $\dfrac{x(x^2-1)}{x^2+1}$ (b) $\dfrac{x(x-1)}{x+1}$ (c) $\dfrac{x-1}{x+1}$ (d) $\dfrac{x^2-1}{x^2+1}$

Ans.: (a)

Solution:

Put $y=vx$ $\Rightarrow\dfrac{dy}{dx}=v+x\dfrac{dv}{dx}$

Thus, we get $v+x\dfrac{dv}{dx}=1+\dfrac{vx}{x}-\dfrac{v^2x^2}{x^2}$ $\Rightarrow v+x\dfrac{dv}{dx}=1+v-v^2=1-v^2$

$$\Rightarrow\frac{dv}{1-v^2}=\frac{dx}{x}\Rightarrow\frac{1}{2}\left[\frac{1}{1-v}+\frac{1}{1+v}\right]dv=\frac{dx}{x}$$

Integrating both sides we get

$$-\frac{1}{2}\ln(1-v)+\frac{1}{2}\ln(1+v)=\ln x+\ln A\Rightarrow\frac{1}{2}\ln\left(\frac{1+v}{1-v}\right)=\ln(xA)\Rightarrow\ln\left(\frac{1+v}{1-v}\right)=\ln(xA)^2$$

Taking antilogy and put $A^2=c$ wet get $\dfrac{1+\frac{y}{x}}{1-\frac{y}{x}}=cx^2$

Now put $x=1$ and $y=0$; $\Rightarrow\dfrac{1+\frac{0}{1}}{1-\frac{0}{1}}=c\times1=c=1$

Thus, $\dfrac{x+y}{x-y}=x^2\Rightarrow x+y=x^3-x^2y\Rightarrow y+x^2y=x^3-x\Rightarrow y(1+x^2)=x(x^2-1)$

$$y=\frac{x(x^2-1)}{1+x^2}$$

Thus (a) is the correct answer.

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Q6. The value of the integral

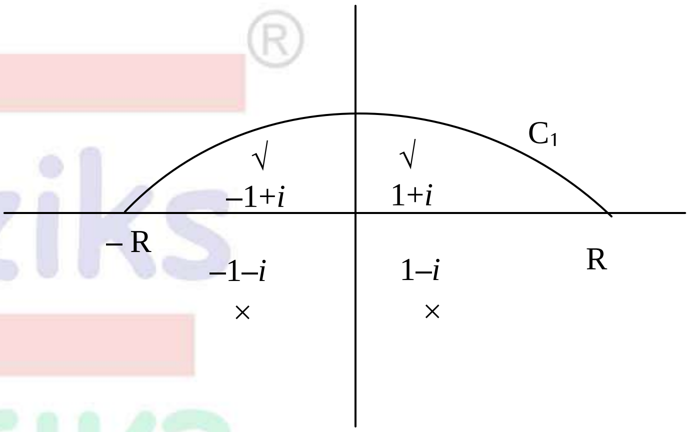
$$\int_0^\infty \frac{dx}{x^4+4} \qquad \text{is}$$

(a) $\frac{\pi}{8}$ (b) $\frac{3\pi}{8}$ (c) 2π (d) $\frac{\pi}{4}$

Ans.: (a)

Solution:

Consider $\int_{-\infty}^{\infty} \frac{dx}{x^4+4}$ and $f(z)=\frac{1}{z^4+4}$



Poles are given by $z^4+4=0$

$\Rightarrow z^4=-4=4e^{\pi}\ \left[e^{\pi}=-1\right]$

$z=4^{1/4}e^{\frac{(2n+1)\pi}{4}}$

$z=\sqrt{2}\,e^{i\left(\frac{2n+1}{4}\right)\pi} \qquad \{n=0,1,2,3\}$

$$z_1=\sqrt{2}\left[\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right]=\sqrt{2}\left[\frac{1}{\sqrt{2}}+\frac{i}{\sqrt{2}}\right]=1+i=\sqrt{2}e^{\frac{i\pi}{4}}$$

$$z_2=\sqrt{2}\left[\cos\frac{3\pi}{4}+i\sin\frac{3\pi}{4}\right]=\sqrt{2}\left[-\frac{1}{\sqrt{2}}+\frac{i}{\sqrt{2}}\right]=-1+i=\sqrt{2}e^{\frac{2\pi}{4}}$$

$z_3=-1-i,\quad z_4=1-i$

z_3 and z_4 are located in the lower half complex plane. Hence, they do not contribute.

Let's calculate residue at $z=z_1$ and $z=z_2$.

z_1 : Residue

$$\frac{1}{\frac{d}{dz}\left(z^4+4\right)\Big|_{z=z_1}}=\frac{1}{4z^3\big|_{z=z_1}}=\frac{1}{4\left(\sqrt{2}\right)^3e^{\frac{3i\pi}{4}}}=\frac{1}{8\sqrt{2}}e^{-\frac{3i\pi}{4}}=\frac{1}{8\sqrt{2}}\left[\cos\frac{3\pi}{4}-i\sin\frac{3\pi}{4}\right]$$

$$=\frac{1}{8\sqrt{2}}\left[\frac{-1}{\sqrt{2}}-\frac{i}{\sqrt{2}}\right]=\frac{-1}{16}(1+i)$$

z_2 : Residue $\frac{1}{4z^3\big|_{z=z_2}}=\frac{1}{4\left(\sqrt{2}e^{\frac{i3\pi}{4}}\right)^3}=\frac{1}{8\sqrt{2}}e^{\frac{-i9\pi}{4}}=\frac{1}{8\sqrt{2}}\left[\frac{1}{\sqrt{2}}-\frac{i}{\sqrt{2}}\right]=\frac{1}{16}[1-i]$

Thus, $\int_C \frac{dz}{z^4+4} = 2\pi i \ \Sigma \text{ Residue } = \int_{-R}^{R} \frac{dx}{x^4+4} + \int_{C_1} f(z)\,dz = 2\pi i \Sigma \text{ Residue}$

Take limit $R \to \infty$ and note that, limit $R \to \infty \int_{C_1} f(z)\,dz = 0$

$$+\int_{-\infty}^{\infty} \frac{dx}{x^4+4} + 0 = 2\pi i\left[\frac{1}{16}(-1-i) + \frac{1}{16}(1-i)\right]$$

$$\int_{-\infty}^{\infty} \frac{dx}{x^4+4} = \frac{2\pi i}{16}\left[-1-i+1-i\right] = \frac{-4\pi i^2}{16} = \frac{\pi}{4}$$

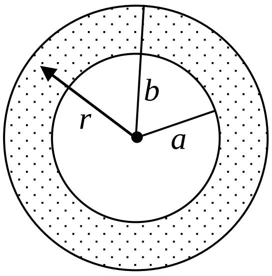
Now $\int_0^{\infty} \frac{dx}{x^4+4} = \frac{1}{2}\int_{-\infty}^{\infty} \frac{dx}{x^4+4} = \frac{1}{2}\frac{\pi}{4} = \frac{\pi}{8}$

Hence the answer is (a)

TIFR-2020 (EMT Question and Solution)

SECTION A-(For both Int. Ph.D. and Ph.D. candidates)

Q1. Consider two concentric spheres of radii a and b, where $a<b$ (see figure). The (shaded) space between thee two spheres is filled uniformly with total charge Q. The electric field at any point between thee two spheres at distance r from the centre is given by



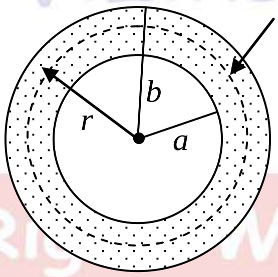
(a) $\frac{Q}{4\pi\varepsilon_0} \frac{r^3-a^3}{r^2\left(b^3-a^3\right)}$ (b) $\frac{Q}{4\pi\varepsilon_0} \frac{1}{r^2}$

(c) $\frac{Q}{4\pi\varepsilon_0}\left(\frac{b}{r^4}-\frac{a}{r^4}\right)^{2/3}$ (d) zero

Ans. : (a)

Solution:

Charge density $\rho = \frac{Q}{\frac{4}{3}\pi\left(b^3-a^3\right)}$



$$\because \oint_S \vec{E}.d\vec{a} = \frac{Q_{enc}}{\varepsilon_0}$$

$$\Rightarrow \left|\vec{E}\right| \times 4\pi r^2 = \frac{1}{\varepsilon_0}\left(\rho \times \frac{4}{3}\pi\left(r^3-a^3\right)\right)$$

$$\Rightarrow \left|\vec{E}\right| \times 4\pi r^2 = \frac{1}{\varepsilon_0}\left(\frac{Q}{\frac{4}{3}\pi\left(b^3-a^3\right)} \times \frac{4}{3}\pi\left(r^3-a^3\right)\right)$$

$$\Rightarrow \left|\vec{E}\right| = \frac{Q}{4\pi\varepsilon_0} \frac{r^3-a^3}{r^2\left(b^3-a^3\right)}$$

Q2. A metallic wire of uniform cross- section and resistance R is bent into a circle of radius a. The circular loop is placed in a magnetic field $\vec{B}(t)$ which is perpendicular to the plane of the wire. This magnetic field is uniform over space, but its magnitude decreases with time at a constant rate k, where

$$k = -\frac{d\left|\vec{B}(t)\right|}{dt}$$

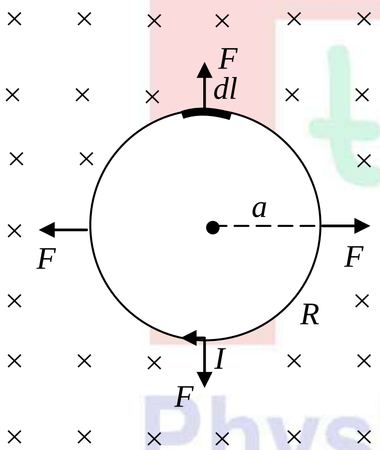
The tension in the metallic wire is

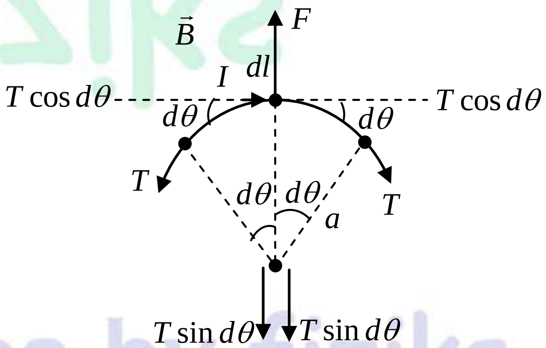
(a) $\frac{\pi a^3 k}{2R}\left|\vec{B}(t)\right|$ (b) $\frac{\pi a^3 k}{R}\left|\vec{B}(t)\right|$

(c) $\frac{2\pi a^3 k}{R}\left|\vec{B}(t)\right|$ (d) zero

Ans. : (b)

Solution:





Magnetic flux $\phi = B(t)\times \pi a^2$.

Induced emf $\varepsilon = -\frac{d\phi}{dt} = -\frac{dB(t)}{dt}\times \pi a^2 = \pi a^2 k$

Induce current $I = \frac{\varepsilon}{R} = \frac{\pi a^2 k}{R}$

Force on elemental length $F = IBdl\sin 90^0 = IBdl$

If element is in equilibrium; $F_{\text{outward}} = F_{\text{inward}}$

$IBdl = 2T\sin d\theta \;\Rightarrow IB(a\times 2d\theta) = 2T(d\theta) \;\Rightarrow T = IBa$

Since $d\theta$ is very small so $\sin d\theta \approx d\theta$ and $2d\theta = \frac{dl}{a}$.

Thus $T = \frac{\pi a^3 k}{R}B(t)$.

Q3. Four students were asked to write down possible forms for the magnetic vector potential $\vec{A}(\vec{x})$ corresponding to a uniform magnetic field of magnitude B along the positive z direction. Three returned correct answers and one returned an incorrect answers. Their answers are reproduced below. Which was the incorrect answer?

(a) $Bx\hat{j}$ (b) $-By\hat{i}$

(c) $\frac{1}{2}\left(Bx\hat{i}-By\hat{j}\right)$ (d) $\frac{1}{2}\left(-By\hat{i}+Bx\hat{j}\right)$

Ans. : (c)

Solution: $\vec{B}=B_0\hat{z}$

(a) $\vec{A}(\vec{x})=Bx\hat{j}$; $\vec{B}=\vec{\nabla}\times\vec{A}=\begin{vmatrix}\hat{x} & \hat{y} & \hat{z}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ 0 & Bx & 0\end{vmatrix}=\hat{x}(0-0)-\hat{y}(0-0)+\hat{z}(B-0)=B\hat{z}$

(b) $\vec{A}(\vec{x})=-By\hat{i}$; $\vec{B}=\vec{\nabla}\times\vec{A}=\begin{vmatrix}\hat{x} & \hat{y} & \hat{z}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ -By & 0 & 0\end{vmatrix}=\hat{x}(0-0)-\hat{y}(0-0)+\hat{z}(0+B)=B\hat{z}$

(c) $\vec{A}(\vec{x})=\frac{1}{2}\left(Bx\hat{i}-By\hat{j}\right)$;

$$\vec{B}=\vec{\nabla}\times\vec{A}=\begin{vmatrix}\hat{x} & \hat{y} & \hat{z}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ \frac{Bx}{2} & -\frac{By}{2} & 0\end{vmatrix}=\hat{x}(0-0)-\hat{y}(0-0)+\hat{z}(0-0)=0$$

(d) $\vec{A}(\vec{x})=\frac{1}{2}\left(-By\hat{i}+Bx\hat{j}\right)$;

$$\vec{B}=\vec{\nabla}\times\vec{A}=\begin{vmatrix}\hat{x} & \hat{y} & \hat{z}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ -\frac{By}{2} & \frac{Bx}{2} & 0\end{vmatrix}=\hat{x}(0-0)-\hat{y}(0-0)+\hat{z}\left(\frac{B}{2}+\frac{B}{2}\right)=B\hat{z}$$

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Q4. The components of the electric and magnetic fields corresponding to a plane electromagnetic field propagating in vacuum satisfy

$$E_x = E_y = -E_z = \frac{|\vec{E}|}{\sqrt{3}} \qquad B_x = -B_y = \frac{|\vec{B}|}{\sqrt{2}} \qquad B_z = 0$$

A unit vector along the direction of propagation of the plane wave is

(a) $\frac{\hat{i}+\hat{j}+2\hat{k}}{\sqrt{6}}$ (b) $-\frac{\hat{i}+\hat{j}+2\hat{k}}{\sqrt{6}}$

(c) $\frac{2\hat{i}-2\hat{j}+\hat{k}}{\sqrt{3}}$ (d) $-\frac{2\hat{i}-2\hat{j}+\hat{k}}{\sqrt{3}}$

Ans. : (b)

Solution:

Let $\vec{E} = \left(E_x\hat{x}+E_y\hat{y}+E_z\hat{z}\right)e^{i\left(\vec{k}.\vec{r}-\omega t\right)}$ and $\vec{B} = \left(B_x\hat{x}+B_y\hat{y}+B_z\hat{z}\right)e^{i\left(\vec{k}.\vec{r}-\omega t\right)}$

$$\because E_x = E_y = -E_z = \frac{|\vec{E}|}{\sqrt{3}}, B_x = -B_y = \frac{|\vec{B}|}{\sqrt{2}}, \qquad B_z = 0$$

$$\vec{E} = \left(\frac{|\vec{E}|}{\sqrt{3}}\hat{x}+\frac{|\vec{E}|}{\sqrt{3}}\hat{y}-\frac{|\vec{E}|}{\sqrt{3}}\hat{z}\right)e^{i\left(\vec{k}.\vec{r}-\omega t\right)} = \frac{|\vec{E}|}{\sqrt{3}}\left(\hat{x}+\hat{y}-\hat{z}\right)e^{i\left(\vec{k}.\vec{r}-\omega t\right)}$$

and $\vec{B} = \left(\frac{|\vec{B}|}{\sqrt{2}}\hat{x}-\frac{|\vec{B}|}{\sqrt{2}}\hat{y}\right)e^{i\left(\vec{k}.\vec{r}-\omega t\right)} = \frac{|\vec{B}|}{\sqrt{2}}\left(\hat{x}-\hat{y}\right)e^{i\left(\vec{k}.\vec{r}-\omega t\right)}$

$$\vec{S} = \frac{1}{\mu_0}\left(\vec{E}\times\vec{B}\right) \Rightarrow \vec{S} \propto \left(\vec{E}\times\vec{B}\right) \propto \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 1 & 1 & -1 \\ 1 & -1 & 0 \end{vmatrix} = \hat{x}(0-1)-\hat{y}(0+1)+\hat{z}(-1-1)$$

$$\Rightarrow \vec{S} \propto \left(-\hat{x}-\hat{y}-2\hat{z}\right) = -\left(\hat{x}+\hat{y}+2\hat{z}\right) \Rightarrow \hat{k} \propto \hat{S} \propto -\frac{\left(\hat{x}+\hat{y}+2\hat{z}\right)}{\sqrt{6}}$$

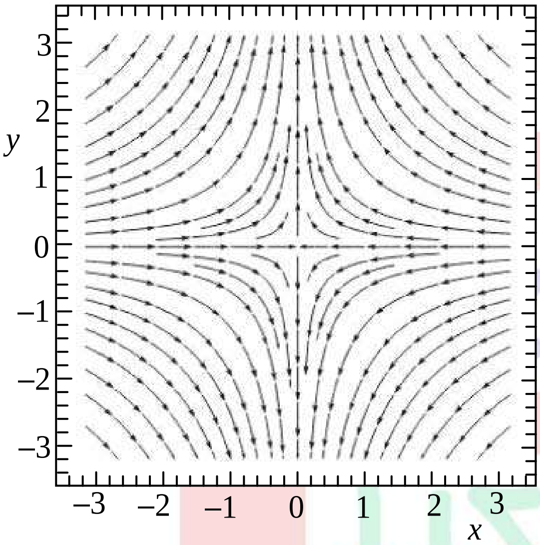
SECTION B- (only for Int.-Ph.D. candidates)

Q5. A two-dimensional electrostatic field is defined as

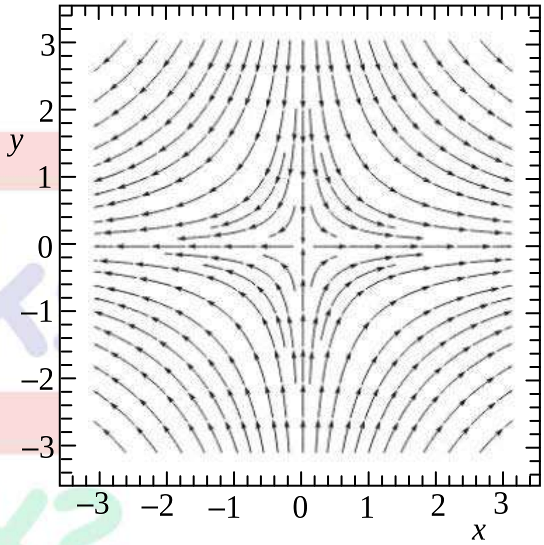
$$\vec{E}(x,y) = -x\hat{i} + y\hat{j}$$

A correct diagram for the lines of force is

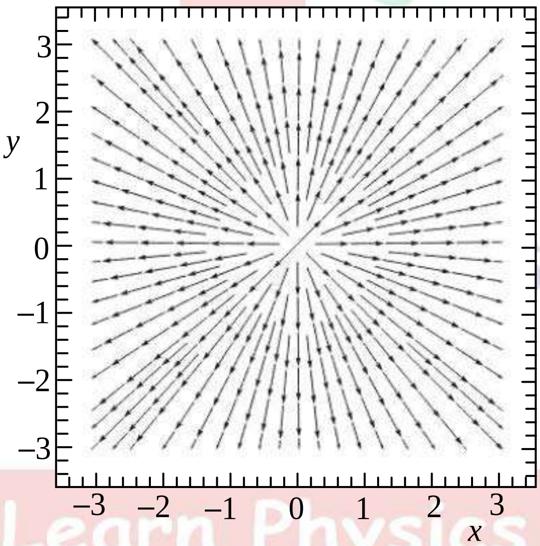
(a)



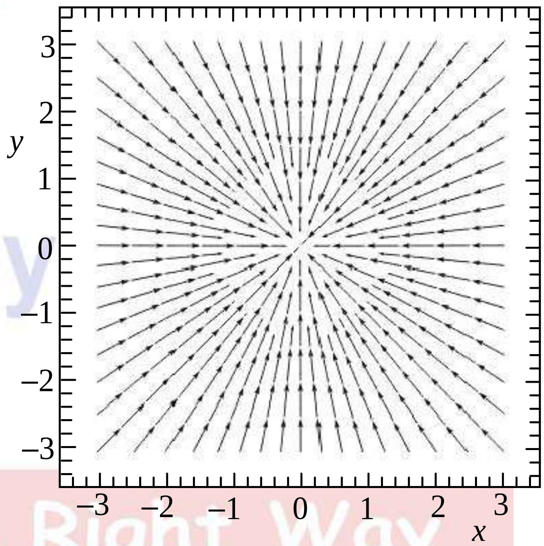
(b)



(c)



(d)



Ans. : (a)

Solution. :

$\because \vec{E}(x,y) = -x\hat{i} + y\hat{j}$

At $\vec{E}(0,0) = 0$, $\vec{E}(1,0) = -\hat{i}$, $\vec{E}(-1,0) = \hat{i}$, $\vec{E}(0,1) = \hat{j}$ and $\vec{E}(0,-1) = -\hat{j}$

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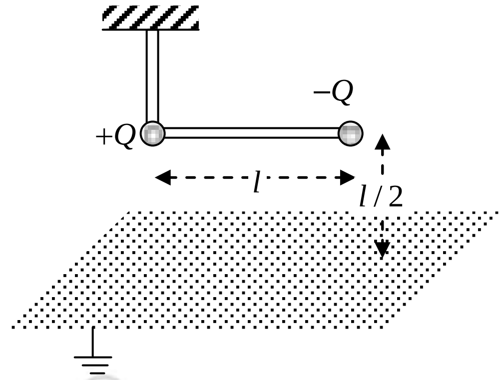
Q6. A light rigid insulating rod of length ℓ is suspended horizontally from a rigid frictionless pivot at one of the ends (see figure). At a vertical distance h below the rod there is an infinite plane conducting plane, which is grounded. If two small, light spherical conductors are attached at the ends of the rod and given charges $+Q$ and $-Q$ as indicated in the figure, the torque on the rod will be



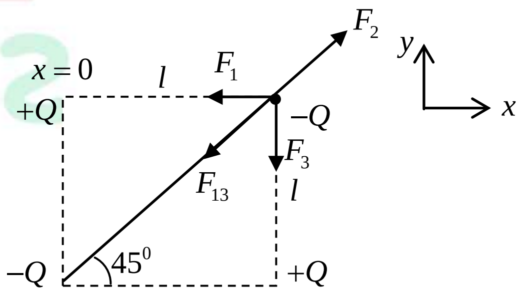
(a) $\frac{Q^2}{4\pi\varepsilon_0 l}\hat{k}$ (b) $-\frac{Q^2}{4\pi\varepsilon_0 l}\hat{k}$

(c) $\frac{\left(4-\sqrt{2}\right)}{16\pi\varepsilon_0}\frac{Q^2}{l}\hat{k}$ (d) $-\frac{\left(4-\sqrt{2}\right)}{16\pi\varepsilon_0}\frac{Q^2}{l}\hat{k}$

Ans.: (d)

Solution:

Force on $-Q$ charge due to other charges are



$$F_1 = F_3 = \frac{Q^2}{4\pi\varepsilon_0 l^2}$$ and

$$F_2 = \frac{Q^2}{4\pi\varepsilon_0\left(\sqrt{2}l\right)^2} = \frac{Q^2}{4\pi\varepsilon_0}\frac{1}{2l^2}.$$

Resultant of F_1 and F_3 is $F_{13} = \sqrt{2}F_1 = \sqrt{2}\frac{Q^2}{4\pi\varepsilon_0 l^2}$

Magnitude of Net Force on $-Q$ charge is $F = F_{13} - F_2 = \frac{Q^2}{4\pi\varepsilon_0 l^2}\left(\sqrt{2}-\frac{1}{2}\right)$

Net Force on $-Q$ charge is

$$\vec{F} = F\cos 45^0\left(-\hat{x}\right) + F\sin 45^0\left(-\hat{y}\right) = \frac{F}{\sqrt{2}}\left(-\hat{x}-\hat{y}\right) = \frac{Q^2}{4\pi\varepsilon_0 l^2}\left(1-\frac{1}{2\sqrt{2}}\right)\left(-\hat{x}-\hat{y}\right)$$

Torque on rod is $\vec{\tau} = \vec{r}\times\vec{F} = \left(l\hat{x}\right)\times\left[\frac{Q^2}{4\pi\varepsilon_0 l^2}\left(1-\frac{1}{2\sqrt{2}}\right)\left(-\hat{x}-\hat{y}\right)\right]$

$$\vec{\tau} = -\frac{Q^2}{4\pi\varepsilon_0 l}\left(1-\frac{1}{2\sqrt{2}}\right)\hat{z} = -\frac{Q^2}{16\pi\varepsilon_0 l}\left(4-\frac{4}{2\sqrt{2}}\right)\hat{z} \Rightarrow \vec{\tau} = -\frac{Q^2}{16\pi\varepsilon_0 l}\left(4-\sqrt{2}\right)\hat{z}$$

Q7. The magnitude vector potential $\vec{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}$ is defined in a region R of space by

$A_x = 5\cos\pi y \qquad A_y = 2 + \sin\pi x \qquad A_z = 0$

in an appropriate unit.

If L be a square loop of wire in the $x-y$ plane, with its end at

$(0,\ 0) \qquad (0,\ 0.25) \qquad (0.25,\ 0.25) \qquad (0.25,\ 0)$

In appropriate unit and it lies entirely in the region R, the numerical value of the flux of the above magnetic field (in the same units) passing through L is

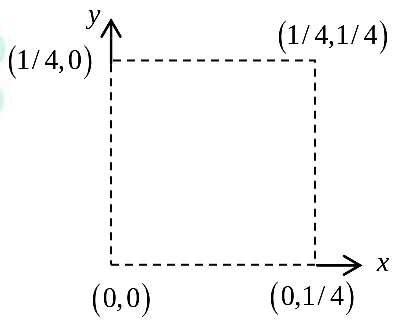
(a) 0.543 (b) 3.31 (c) −0.75 (d) *zero*

Ans.: (a)

Solution. :

$$\vec{B} = \vec{\nabla}\times\vec{A} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 5\cos\pi y & 2+\sin\pi x & 0 \end{vmatrix} = \hat{x}(0-0) - \hat{y}(0-0) + \hat{z}(\pi\cos\pi x + 5\pi\sin\pi y)$$

$$\Rightarrow \vec{B} = \hat{z}(\pi\cos\pi x + 5\pi\sin\pi y)$$



The flux of the magnetic field is

$$\phi = \int_S \vec{B}.d\vec{a} = \int_0^{1/4}\int_0^{1/4} (\pi\cos\pi x + 5\pi\sin\pi y)\hat{z}.(dxdy\hat{z})$$

$$\phi = \int_{y=0}^{1/4}\left[\frac{\pi\sin\pi x}{\pi} + 5\pi x\sin\pi y\right]_{x=0}^{1/4} dy$$

$$\Rightarrow \phi = \int_{y=0}^{1/4}\left[\sin\frac{\pi}{4} + \frac{5}{4}\pi\sin\pi y\right]dy$$

$$\Rightarrow \phi = \left[\frac{y}{\sqrt{2}} + \frac{5}{4}\pi\left(-\frac{\cos\pi y}{\pi}\right)\right]_{y=0}^{1/4} \Rightarrow \phi = \left[\frac{1}{4\sqrt{2}} - \frac{5}{4}\times\frac{1}{\sqrt{2}} + \frac{5}{4}\right] \Rightarrow \phi = \frac{1}{4\sqrt{2}}\left[1 - 5 + 5\sqrt{2}\right]$$

$$\Rightarrow \phi = \frac{1}{4\sqrt{2}}\left[1 + 5\left(\sqrt{2}-1\right)\right] \Rightarrow \phi = \frac{1}{4\sqrt{2}}[1 + 5\times 0.414] = \frac{3.07}{4\sqrt{2}} = 0.543$$

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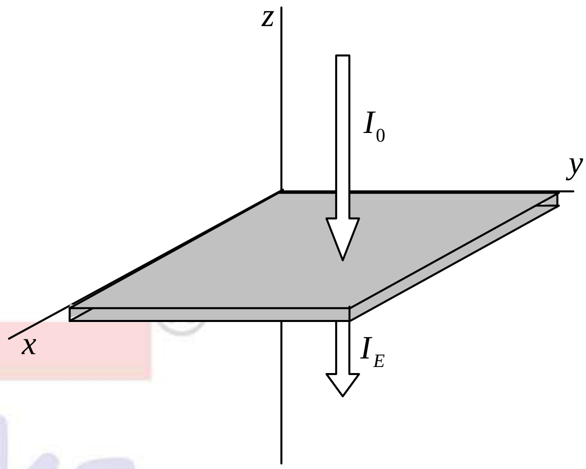
SECTION B-(Only for Ph.D. candidates)

Q8. Consider the following situation.

An infinite plane metallic plate of thickness $1.8\,cm$ is placed along the $x-y$ plane, with z axis normal to the sheet (see figure).



A plane radio wave of intensity I_0 and frequency $29.5\,MHz$ propagates in vacuum along the negative z-axis and strikes the metal foil at normal incidence. If the metal of the foil has conductivity $5.9\,\Omega^{-1}m^{-1}$ and magnetic permeability $\mu \simeq 1$, the intensity I_E of the emergent wave will be approximately

(a) $0.26\,I_0$ (b) $0.51 I_0$

(c) $0.29\times10^{-7}\,I_0$ (d) $2.08\times10^{-4}\,I_0$

Ans.: (a)

Solution:

Intensity inside conductor $I = I_0 e^{-2\kappa z}$ where $\kappa = \sqrt{\pi\sigma\mu f} = \sqrt{\pi\sigma(\mu_0\mu_r)f}$.

Thus $\kappa = \sqrt{3.14\times5.9\times(4\times3.14\times10^{-7})29.5\times10^{6}} \approx 26.3$

$I = I_0 e^{-2\times26.3\times1.8\times10^{-2}} = I_0 e^{-0.943} = 0.389 I_0$

Best option is (a).

TIFR-2020 (Quantum Mechanics Question and Solution)

SECTION A-(For both Int. Ph.D. and Ph.D. candidates)

Q1. The wave function of a particle subjected to a three-dimensional spherically-symmetric potential $V(r)$ is given by

$$\psi(\vec{x}) = (x+y+3z)\,f(r)$$

the expectation value for the operator $\vec{L}^2$ for this state is

(a) $\hbar^2$ (b) $2\hbar^2$ (c) $5\hbar^2$ (d) $11\hbar^2$

Ans.: (b)

Solution:

The wave function of a particle subjected to a three-dimensional spherically symmetric potential $\psi(r)$ is given by

$$\psi(\vec{x}) = (x + y + 3z) f(r)$$

writing $Y_1^{\pm1,0}$ state using spherical harmonies in Cartesian co-ordinates, we, have,

$$Y_1^{\pm1} = \pm\left(\frac{3}{4\pi}\right)^{1/2} \frac{x \pm iy}{\sqrt{2}r} \; ; \; Y_1^0 = \left(\frac{3}{4\pi}\right)^{1/2} \left(\frac{z}{r}\right) \Rightarrow \frac{z}{r} = \left(\frac{4\pi}{3}\right)^{1/2} Y_1^0$$

Simplifying the value of x & y from Y_1^{+1} as follows

$$Y_1^{+1} = -\left(\frac{3}{8\pi}\right)^{1/2} \frac{(x+iy)}{r} \; ; \; Y_1^{-1} = \left(\frac{3}{8\pi}\right)^{1/2} \frac{(x-iy)}{r}, \qquad (1a, 1b)$$

on addition, of eq $(1a)$ and eq $(1b)$, we get

$$Y_1^{+1} + Y_1^{-1} = \sqrt{\frac{3}{8\pi}}(-2)\left(\frac{iy}{r}\right) \Rightarrow \frac{y}{r} = i\sqrt{\frac{2\pi}{3}}\left(Y_1^{+1} + Y_1^{-1}\right)$$

on subtracting eq $(1a)$ from eq $(1b)$, we obtain

$$Y_1^{-1} - Y_1^{+1} = \sqrt{\frac{3}{8\pi}} \frac{2x}{r} \Rightarrow \frac{x}{r} = \sqrt{\frac{2\pi}{3}}\left(Y_1^{-1} - Y_1^{+1}\right)$$

Substituting the value of x, y and z in the wave function, we obtain,

$$\psi = rf(r)\left(\sqrt{\frac{2\pi}{3}}\left(Y_1^{-1} - Y_1^{+1}\right) + i\sqrt{\frac{2\pi}{3}}\left(Y_1^{+1} + Y_1^{+1}\right) + 3\sqrt{\frac{4\pi}{3}} Y_1^0\right)$$

$$= rf(r)\left[(1+i)\sqrt{\frac{2\pi}{3}} Y_1^{-1} + (1-i)\sqrt{\frac{2\pi}{3}} Y_1^{+1} + 3\sqrt{\frac{4\pi}{3}} Y_1^0\right]$$

Expressing ψ in term of eigenstates $|\ell, m\rangle$

$$|\psi\rangle = N\left[(1+i)\sqrt{\frac{2\pi}{3}}|1,-1\rangle - (1-i)\sqrt{\frac{2\pi}{3}}|1,+1\rangle + 3\sqrt{\frac{4\pi}{3}}|1,0\rangle\right]$$

where N is the normalization constant

$$\langle\psi|\psi\rangle = N^2\left[(1-i)(1+i)\frac{2\pi}{3}\langle 1,-1|1,-1\rangle + (1-i)(1+i)\left(\frac{2\pi}{3}\right)\langle 1,1|1,1\rangle + 9.\frac{4\pi}{3}\langle 1,0|1,0\rangle\right] = 1$$

or

$$N^2\left[\frac{4\pi}{3} + \frac{4\pi}{3} + 9.\frac{4\pi}{3}\right] = 1 \Rightarrow N = \sqrt{\frac{3}{4\pi}} \cdot \frac{1}{\sqrt{11}}$$

The normalized wave function is given by.

$$|\psi\rangle = \frac{(1+i)}{\sqrt{22}}|1-1\rangle - \frac{(1-i)}{\sqrt{22}}|1,1\rangle + \frac{3}{\sqrt{11}}|1,0\rangle$$

The expectation value of L^2 is given by

$$\left\langle L^2\right\rangle=\left\langle\psi\left|L^2\psi\right.\right\rangle=\left\langle\psi\right|\left(\frac{1+i}{\sqrt{22}}L^2\left|1,-1\right\rangle-\frac{(1-i)}{\sqrt{22}}L^2\left|1,1\right\rangle+\frac{3}{\sqrt{11}}L^2\left|1,0\right\rangle\right)$$

$$=\left\langle\psi\right|\left(\frac{1+i}{\sqrt{22}}\left(2\hbar^2\right)\left|1,-1\right\rangle-\frac{(1-i)}{\sqrt{22}}\left(2\hbar^2\right)\left|1,1\right\rangle+\frac{3}{\sqrt{11}}\left(2\hbar^2\right)\left|1,0\right\rangle\right)$$

$$\left\langle L^2\right\rangle=\left\langle\psi\right|\left(\left|\left(2\hbar^2\right)\right|\frac{(1+i)\left|1,-1\right\rangle}{\sqrt{22}}-\frac{(1-i)\left|1,1\right\rangle}{\sqrt{22}}+\frac{3\left|1,-0\right\rangle}{\sqrt{11}}\right)=\left(2\hbar^2\right)\left\langle\psi\middle|\psi\right\rangle=2\hbar^2$$

Q2. A fermion of mass m, moving in two dimensions, is strictly confined inside a square box of side ℓ. The potential inside is zero. A measurement of the energy of the fermion yields the result

$$E=\frac{65\pi^2\hbar^2}{2m\ell^2}$$

The degeneracy of this energy state is

(a) 2 (b) 4 (c) 8 (d) 16

Ans.: (c)

Solution:

A fermions of mass m, moving in two dimensions, is strictly confined inside a square box of side ℓ. The potential inside is zero. The energy of the fermions in this system is

$$E_{n_x,n_y}=\left(n_x^2+n_y^2\right)\frac{\pi^2\hbar}{2m\ell^2}$$

A measurement of the energy of the fermions. Yields the result.

$$E_n=\frac{65\pi^2\hbar^2}{2mL^2}$$

Comparing the two energies, we have.

$$E_{n_x,n_y}=\left(n_x^2+n_y^2\right)\frac{\pi^2\hbar^2}{2mL^2}=\frac{65\pi^2\hbar^2}{2mL^2}$$

or

$$n_x^2+n_y^2=65\Rightarrow\begin{cases}\left(n_x,n_y\right)=(7,4)\\\left(n_x,n_y\right)=(4,7)\\\left(n_x,n_y\right)=(8,1)\\\left(n_x,n_y\right)=(1,8)\end{cases}$$

Thus, the degeneracy in the energy state is 4. Since the particle is fermions. The degeneracy in the energy state is 8.

Q3. The momentum operator

$$i\hbar\frac{d}{dx}$$

acts on a wavefunction $\psi(x)$. This operator is Hermitian

(a) provided the wavefunction $\psi(x)$ is normalized

(b) provided the wavefunction $\psi(x)$ and derivate $\psi'(x)$ are continuous everywhere

(c) provided the wavefunction $\psi(x)$ vanishes as $x\to\pm\infty$

(d) by its very definition

Ans. : (c)

Solution:

The momentum operator

$$i\hbar\frac{d}{dx}$$

acts on a wave function. This operator is Hermitian when $\psi(x)$ is finite *i.e.*, the wave function $\psi(x)$ vanishes as $x\to\pm\infty$.

SECTION B- (only for Int.-Ph.D. candidates)

Q4. A particle of mass m is confined inside a box with boundaries at $x=\pm L$. The ground state and the first excited state of this particle are E_1 and E_2 respectively.

Now a repulsive delta function potential $\lambda\delta(x)$ is introduced at the centre of the box where the constant λ satisfies

$$0<\lambda\ll\frac{1}{32m}\left(\frac{h}{L}\right)^2$$

If the energies of the new ground state and the new first excited state be denoted as E_1' and E_2' respectively, it follows that

(a) $E_1'>E_1, E_2'>E_2$

(b) $E_1'=E_1, E_2'=E_2$

(c) $E_1'>E_1, E_2'=E_2$

(d) $E_1'=E_1, E_2'>E_2$

Ans.: (c)

Solution:

A particle of mass m is contained inside a box with boundaries at $x=\pm L$. The ground state and first excited wave function. And their corresponding energies are given by.

$$\psi_1(x)=\sqrt{\frac{1}{L}}\cos\frac{\pi x}{2L}\ ;\ E=E_1$$

and

$$\psi_2(x)=\sqrt{\frac{1}{L}}\sin\frac{\pi x}{2L}\ ;\ E=E_2.$$

Now a repulsive delta function potential $\lambda\delta(x)$ is introduced at the centre of the box where the constant λ is given, $0<\lambda<<\frac{1}{32m}\left(\frac{h}{L}\right)^2$

The perturbed Hamiltonian of the system is $H'(x)=\lambda\delta(x)$

The correction is the ground state energy is given by.

$$E_1^0=\langle\psi_1(x)|H'(x)|\psi_1(x)\rangle=\int\psi_1^*(x)\psi_1(x)H'(x)dx=\int_{-L}^{L}\frac{1}{L}\lambda\cos^2\frac{\lambda x}{2L}\cdot\delta(x)dx$$

$$=\frac{\lambda}{L}\int_{-L}^{L}\cos^2\frac{\lambda x}{2L}\delta(x)dx=\frac{\lambda}{L}\cos^2\frac{\pi}{2L}.=\frac{\lambda}{L}$$

Where we have used $\int F(x)\delta(x-a)dx=F(a)$.

The new ground state energy of the particle is

$$E_1^1=E_1+E_1^{(1)}=E_1+\frac{\lambda}{L}>E_1$$

The correction in the first excited state energy of the particle is

$$E_2^{(1)}=\langle\psi_2(x)|H'(x)|\psi_2(x)\rangle=\int\psi_2^*(x)\psi_2(x)H'(x)d$$

$$=\int_{-L}^{L}\frac{1}{L}\sin^2\frac{\pi x}{2L}\lambda\delta(x)dx=\frac{\lambda}{L}\int_{-L}^{L}\sin^2\frac{\pi x}{2L}\delta(x)dx=0\,.$$

Where, we have used, $\int F(x)\delta(x-a)dx=F(a)$

Q5. Three non-interacting particles whose masses are in the ratio $1:4:16$ are placed together in the same harmonic oscillator potential $V(x)$.

The degeneracies of the first three energy eigenstates (ordered by increasing energy) will be

(a) 1,1,1 (b) 1,1,2 (c) 1,2,1 (d) 1,2,2

Ans.: (b)

Solution:

The masses of the there non interacting identical particle are in the ratio $1:4:16$. The masses of these three particles are.

$m_i = m,\ m_2 = 4m,$ and $m_3 = 16m$.

These particles are placed in same harmonic oscillator potential $V(x)$. The potential energy of there particle are given by

$$V(x) = V_1(x_1) + V_2(x_2) + V_3(x_3) = \frac{1}{2}m_1\omega^2 x_1^2 + \frac{1}{2}m_2\omega^2 x_2^2 + \frac{1}{2}m_3\omega^2 x_3^2$$

$$= \frac{1}{2}m\omega^2 x_1^2 + \frac{1}{2}m(2\omega)^2 x_2^2 + \frac{1}{2}m(4\omega)^2 x_3^2$$

$$= \frac{1}{2}m\omega_1^2 x_1^2 + \frac{1}{2}m\omega_2^2 x_2^2 + \frac{1}{2}m\omega_3^2 x_3^2$$

Where, we have defined,

$\omega_1 = \omega;\ \omega_2 = 2\omega; \omega_3 = 4\omega.$

The Hamiltonian for the given system of particle.

$$H = \frac{p^2}{2m} + V(x) = \frac{-\hbar^2}{2m}\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}\right) + \frac{1}{2}m\omega_1^2 x_1^2 + \frac{1}{2}m\omega_2^2 x^2 + \frac{1}{2}m\omega_3^2 x^2$$

$$H = \frac{-\hbar^2}{2m}\frac{\partial^2}{\partial x_1^2} + \frac{1}{2}m\omega_1^2 x_1^2 - \frac{\hbar^2}{2m}\frac{\partial^2}{\partial x_2^2} + \frac{1}{2}m\omega_2^2 x_2^2 - \frac{\hbar^2}{2m}\frac{\partial^2}{\partial x_3^2} + \frac{1}{2}m\omega_3^2 x_3^2$$

$$= H_1(x_1) + H(x_2) + H(x_3)$$

Where, we have defined

$$H_1(x_1) = \frac{-\hbar^2}{2m}\frac{\partial^2}{\partial x_1^2} + \frac{1}{2}m\omega_1^2 x_1^2;\ H_2(x_2) = \frac{-\hbar^2}{2m}\frac{\partial^2}{\partial x_2^2} + \frac{1}{2}m\omega_2^2 x_2^2$$

$$H_3(x_3) = \frac{-\hbar^2}{2m}\frac{\partial^2}{\partial x_3^2} + \frac{1}{2}m\omega_3^2 x_3^2.$$

The energy corresponding to H is given by

$$E_{n_1,n_2,n_3} = E_1 + E_2 + E_3 = \left(n_1 + \frac{1}{2}\right)\hbar\omega_1 + \left(n_2 + \frac{1}{2}\right)\hbar\omega_2 + \left(n_3 + \frac{1}{2}\right)\hbar\omega_3$$

$$= \left(n_1 + \frac{1}{2}\right)\hbar\omega + \left(n_2 + \frac{1}{2}\right)\hbar(2\omega) + \left(n_3 + \frac{1}{2}\right)\hbar(3\omega)$$

$$= \left(n_1 + \frac{1}{2} + 2n_2 + 1 + 4n_3 + 2\right)\hbar\omega = \left(n_1 + 2n_2 + 4n_3 + \frac{7}{2}\right)\hbar\omega.$$

The ground state energy of the system is

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$$E_{0,0,0}=\left(0+0+0+\frac{7}{2}\right)\hbar\omega=\frac{7}{2}\hbar\omega;\ g_1=1$$

The first excited state energy of the system is

$$E_{1,0,0}=\left(1+\frac{7}{2}\right)\hbar\omega=\frac{9}{2}\hbar\omega;\ g_2=1$$

The second excited state energy of the system is

$$E_{0,1,0}=\left[0+2+0+\frac{7}{2}\right]\hbar\omega=\frac{11}{2}\hbar\omega;E_{2,0,0}=\left[2+0+0+\frac{7}{2}\right]\hbar\omega=\frac{11}{2}\hbar\omega;\ g_3=2$$

The Degeneracy's of the first three energies (in increasing order) is given by

$$g_1,g_2,g_3=1,1,\ 2.$$

Here degeneracy's is denoted by g_i^s.

SECTION B-(Only for Ph.D. candidates)

Q6. A particle of mass m is placed in one dimensional harmonic oscillator potential

$$V(x)=\frac{1}{2}m\omega^2x^2$$

At $t=0$, its wavefunction is $\psi(x)$. At $t=2\pi/\omega$ its wavefunction will be

(a) $\psi(x)$ (b) $-\psi(x)$ (c) $-\pi\psi(x)$ (d) $\frac{2\pi}{\omega}\psi(x)$

Ans.: (b)

Solution:

A particle of mass m is placed in a $1-d$ Harmonic potential

$$v(x)=\frac{1}{2}m\omega^2x^2.$$

The wave function of the particle at time $t=0$ is given by

$$\psi(x,0)=\psi(x)$$

The wave function of the particle at time t is given by

$$\psi_0(x,t)=\psi_0(x)e^{-iE_0t/\hbar}=\psi_0(x)e^{-i(\hbar\omega/2\hbar)(2\pi/\omega)}$$

$$=\psi_0(x)e^{-i\pi}=-\psi_0(x),$$

Where, we have used, E is the ground state energy of the particle.

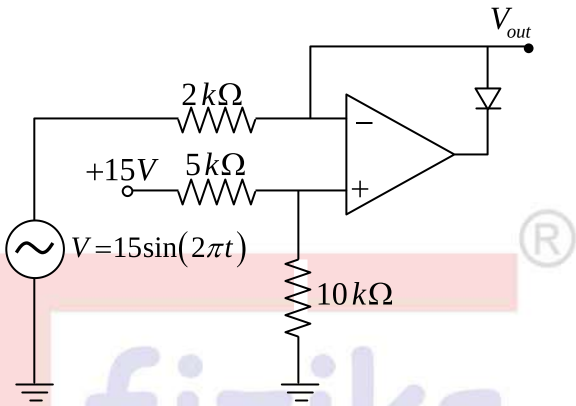
$$E=\frac{\hbar\omega}{2}.$$

One would obtain same result for other values of energies as well.

TIFR-2020 (Electronics Question and Solution)

SECTION A-(For both Int. Ph.D. and Ph.D. candidates)

Q1.



In the above circuit, which of the following is the maximum value, in Volts, of voltage at V_{out} ?

(a) 10 (b) 15 (c) 0 (d) 5

Ans. : (a)

Solution:

When diode is off output will be zero.

When diode is on output is equal to voltage at non-inverting input.

Voltage at non-inverting input $v_1 = \dfrac{10k}{10k+5k} \times 15V = 10V$.

Q2. A badly-designed voltmeter is modelled as an ideal voltmeter with a large resistor (R) and a large capacitor (C) connected in parallel to it. Given this information, which of the following statements describes what happens when this voltmeter is connected to a DC voltage source with voltage V and internal resistance $r(r \ll R)$?

(a) The reading on the voltameter rises slowly and becomes steady at a value slightly less than V

(b) The reading on the voltameter starts at a value slightly less than V and slowly falls to zero.

(c) The reading on the voltameter rises slowly to maximum value close to V and then slowly goes to zero.

(d) The reading on the voltameter reads zero even when connected to the voltage source.

Ans. : (a)

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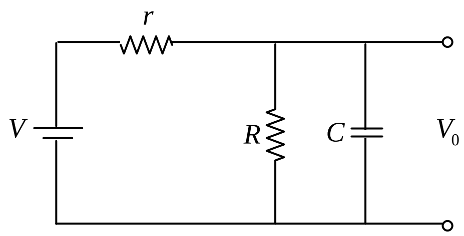
Solution:



In steady State capacitor is open

$$V_0 = \left(\frac{R}{r+R}\right)V \quad < V$$

Q3. An OR gate, a NOR gate and an XOR gate are to be constructed using only NAND gates. If the minimum number of NAND gates needed to construct OR, NOR and XOR gates is denoted $n(\text{OR}), n(\text{NOR})$ and $n(\text{XOR})$ respectively, then

(a) $n(\text{NOR}) = n(\text{XOR}) > n(\text{OR})$

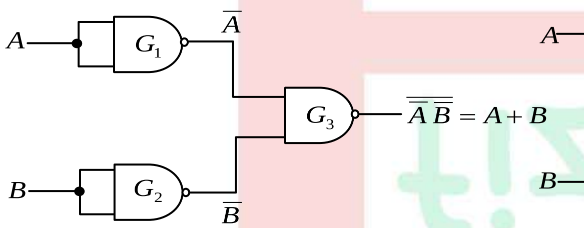
(b) $n(\text{NOR}) = n(\text{XOR}) = n(\text{OR})$

(c) $n(\text{NOR}) > n(\text{XOR}) > n(\text{OR})$

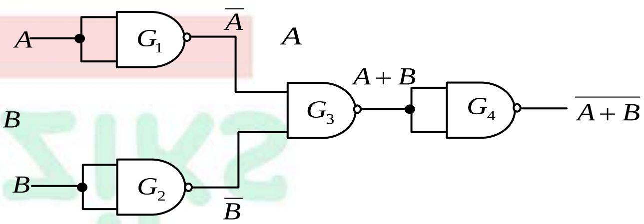
(d) $n(\text{NOR}) < n(\text{XOR}) = n(\text{OR})$

Ans. :(a)

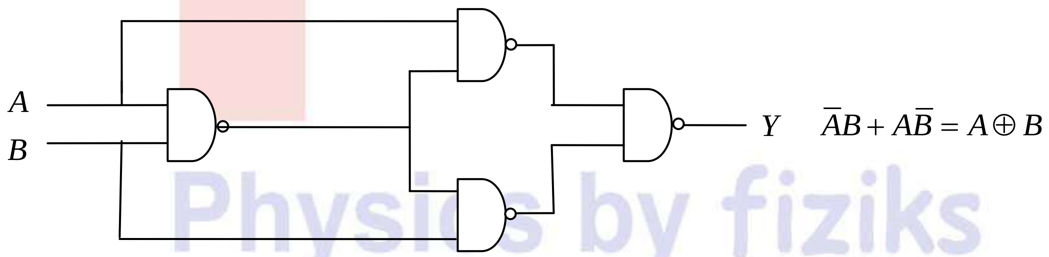
Solution:



(a) Three NAND gate used as an OR gate



(b) Four NAND gate used as a NOR gate

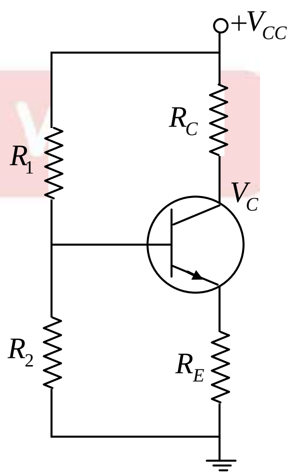


(c) Four NAND gate used as a XOR gate

SECTION B- (only for Int.-Ph.D. candidates)

Q4. The circuit shown represents a typical voltage-divider bias circuit for a transistor. Assume that resistance values and voltage values are typical for using the transistor as an amplifier.

Which of the following changes in the circuit would result in an increase in the collector voltage V_C?



(a) R_2 is decreased slightly

(b) R_2 is increased slightly

(c) R_C is decreased slightly

(d) R_C is increased slightly

Ans.: (a)

Solution:

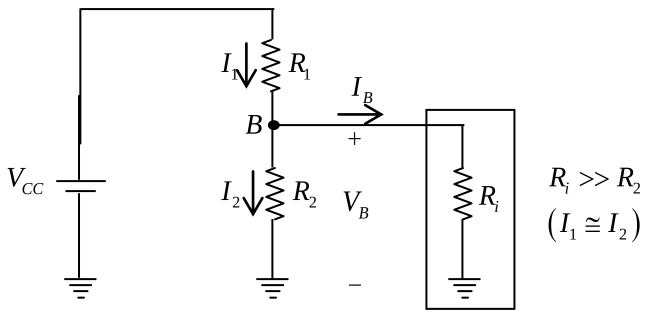


Figure: Redrawing the input side of the network.

Under this approximation $V_B = I_2R_2 = \dfrac{V_{CC}R_2}{R_1+R_2} = \dfrac{V_{CC}}{\dfrac{R_1}{R_2}+1}$ as $R_2 \downarrow, \dfrac{R_1}{R_2} \uparrow \Rightarrow V_B \downarrow$

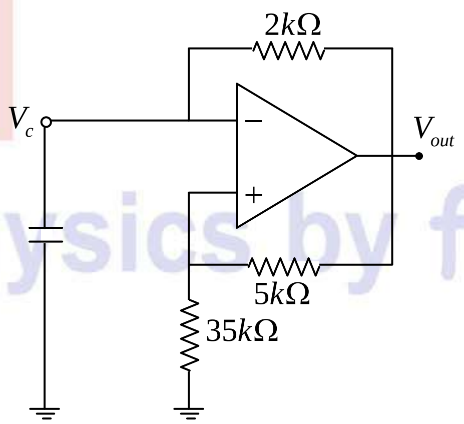
Once V_B is determined, the level of V_E can be calculated from $V_E = V_B - V_{BE}$.

As $V_B \downarrow \ \Rightarrow V_E \downarrow \Rightarrow I_E \approx I_C = \dfrac{V_E}{R_E} \downarrow$.

$\because V_C = V_{CC} - I_C R_C$, as $I_C \downarrow \ \Rightarrow V_C \uparrow$.

SECTION B-(Only for Ph.D. candidates)

Q5. The circuit sketched below is called a relaxation oscillator.



For the parameters indicated in the figure, the ratio of the maximum voltage at V_{out} to the maximum voltage at V_C is

(a) $\dfrac{1}{8}$ (b) $\dfrac{1}{7}$ (c) $\dfrac{2}{7}$ (d) $\dfrac{1}{4}$

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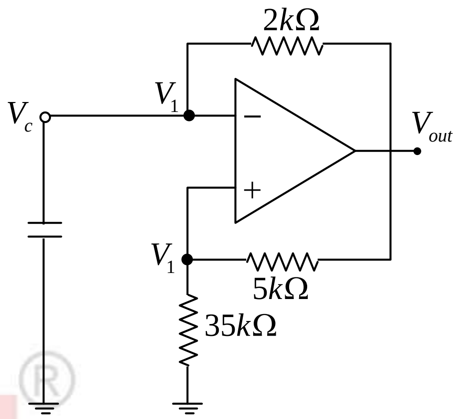
Ans. : no option matches

Solution:



It's a square wave generator with output limited between $\pm V_{sat}$.

So $V_{\text{out,max}} = V_{sat}$

Maximum value at V_C is $V_1 = \left(\frac{35k}{35k+5k}\right)V_{sat} = \frac{7}{8}V_{sat}$.

So $\frac{V_{\text{out,max}}}{V_{C,\max}} = \frac{V_{sat}}{\frac{7}{8}V_{sat}} = \frac{8}{7}$

TIFR-2020 (Solid State Physics Questions and Solution)

SECTION B- (only for Int.-Ph.D. candidates)

Q1. A beam of X-rays is incident upon a powder sample of a material which forms simple cubic crystals of lattice constant $5.5\,\overset{o}{A}$. The maximum wavelength of the X-rays which can produce diffraction from the planes with Miller indices $(0,0,5)$ is

(a) $2.2\,\overset{o}{A}$ (b) $55.0\,\overset{o}{A}$ (c) $1.1\,\overset{o}{A}$ (d) $27.5\,\overset{o}{A}$

Ans.: (a)

Solution:

Bragg's law is

$$2d\sin(\theta) = n\lambda$$

For the maximum wavelength corresponds to

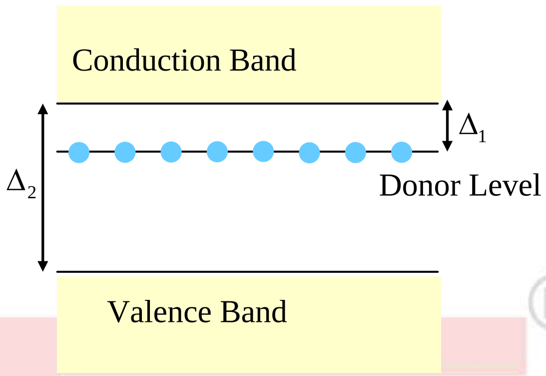
$$\sin(\theta) = \sin(90^0) = 1 \text{ and } n = 1$$

The wavelength of the x-ray is

$$\lambda = 2d = \frac{2a}{\sqrt{h^2+k^2+l^2}} = \frac{2\times 5.5\,\overset{0}{\text{A}}}{\sqrt{0^2+0^2+5^2}} = \frac{11\,\overset{0}{\text{A}}}{5} = 2.2\,\overset{0}{\text{A}}$$

Thus, correct option is (a).

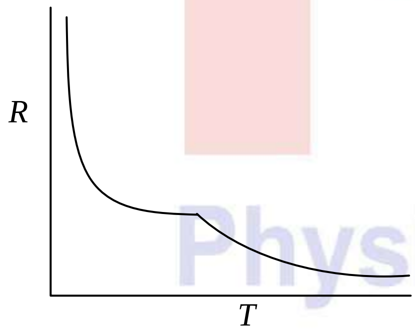
Q2. A semiconductor with donor impurities can be thought in terms of a filled valence band, a filled donor level and an empty valence band at $T = 0$, as shown in the figure below



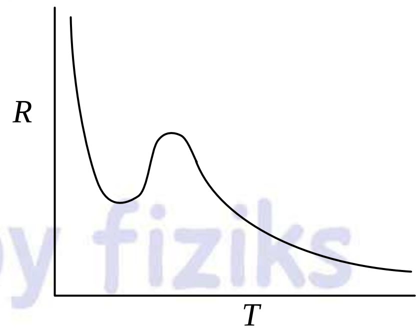
If the band gap between donor level and conduction band is Δ_1 and that between conduction and valence band is Δ_2 where $\Delta_2 \gg \Delta_1$, which of the following figures depict the qualitative features of the resistance (R)- vs-temperature (T) graph of the semi-conductor?

(Assume temperature-independent scattering rates and a flat density of states for the bands.)

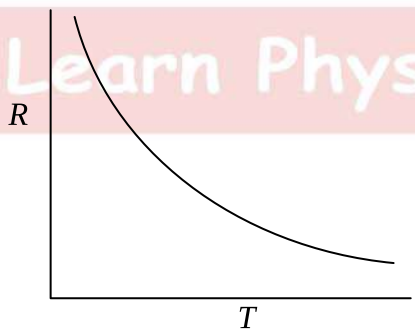
(a)



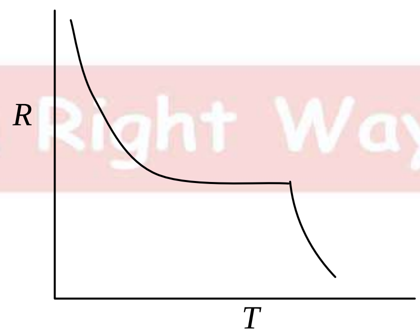
(b)



(c)



(d)



Ans.: (a)

Solution:

The electrical conductivity of an n-type semiconductor can be calculated from

$$\sigma = en\mu_n \cong eN_d^+\mu_n$$

Where, N_d^+ is the ionized donor concentration and μ_n is the mobility of the electron which depends on the scattering of the electrons. In the question is mention to assume temperature-independent scattering rates, it means μ_n is temperature independent.

The sample resistance can be written in term of resistivity ρ, length (L) and cross-section area of the sample through which current is flowing

$$R = \rho\frac{L}{A} = \frac{1}{\sigma}\frac{L}{A} = \frac{1}{eN_d^+}\frac{L}{A} \quad \Rightarrow \quad R \propto \frac{1}{N_d^+}$$

At absolute zero degree $(T = 0K)$, all electrons are un their lowest possible energy state; that is, for n-type semiconductor, each donor state must contain an electron, therefore $N_d^+ = 0$, it means resistance will be infinite.

As the temperature increase electrons from the donor levels start moving into the conduction band, as a result N_d^+ increases and R decreases. This region is also called partial ionization region. Since Δ_1 is of the order of few meV only, therefore at moderate but below room temperature the donor's states are essentially completely ionized and almost all donor impurity atoms have donated an electron to the conductions band. In these region R remains constant, this region is called extrinsic region.

At higher temperature, electron-hole pairs are thermally generated and these electrons from the valence band start moving into the conduction band. Since the intrinsic carrier concentration is very strong function of temperature, therefore the intrinsic carrier concentration n_i may begin to dominate. The semiconductor will eventually lose its extrinsic nature, and in this region, resistance falls quickly but never becomes zero.

Therefore, the correct variation of the R vs T is option (a) not the option (d).

TIFR-2020 (Nuclear Physics Questions and Solution)

SECTION B- (only for Int.-Ph.D. candidates)

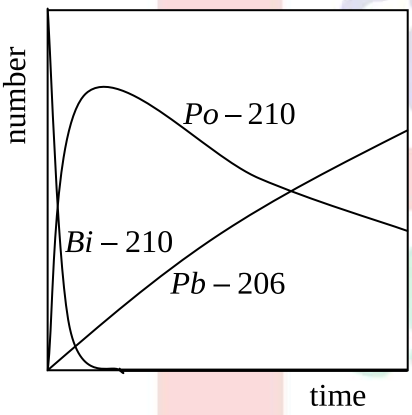
Q1. Consider the nuclear decay chain of radio-Bismuth to Polonium to Lead, i.e.

$$^{219}_{83}\mathrm{Bi} \rightarrow ^{210}_{84}\mathrm{Po} \rightarrow ^{206}_{82}\mathrm{Pb}$$

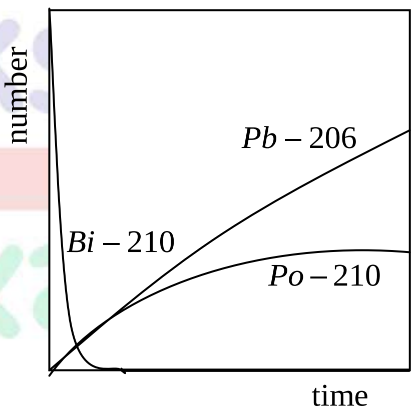
where Pb-206$\left(^{206}_{82}\mathrm{Pb}\right)$ is a stable nucleus, and $\mathrm{Bi}\text{-}210\left(^{219}_{82}\mathrm{Bi}\right)$ and $\mathrm{Po}\text{-}206\left(^{210}_{84}\mathrm{Po}\right)$ are radioactive nuclei with half lives of about 5 days and 138 days respectively.

If we start with a sample of pure $\mathrm{Bi}\text{-}210\left(^{219}_{82}\mathrm{Bi}\right)$, then a possible graph for the time evolution of the number of nuclei of these three species will be

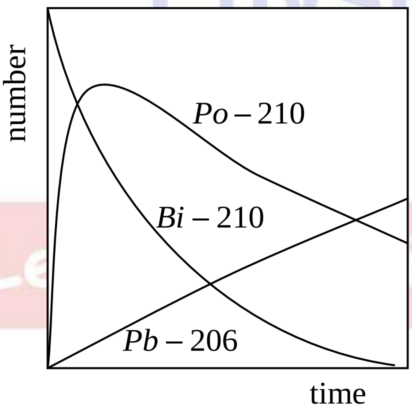
(a)



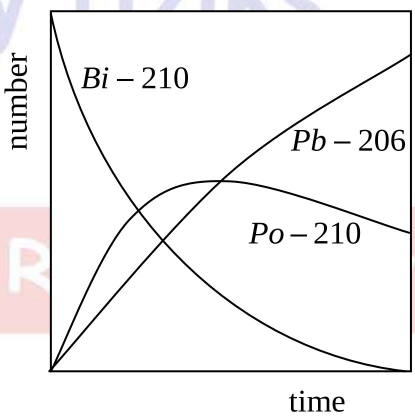
(b)



(c)



(d)



Ans.: (a)

Solution:

This is a problem of successive decay. As $t_{1/2}(Bi) << t_{1/2}(Po)$, so decay and growth of $Bi-210$ and $Po-210$ will be very quick. Number of nuclei of $Po-210$ will increase upto a maximum value and then start decreasing. The number of nuclei of $Pb-206$ will increase continuously upto the time when their number will become equal to N_0.

$$N_{Bi} = N_0 e^{-\lambda_1 t}; \quad N_{Po} = \frac{N_0 \lambda_1}{\lambda_2 - \lambda_1}\left[e^{-\lambda_1 t} - e^{-\lambda_2 t}\right]$$

$$N_{Pb} = N_0 - N_{Bi} - N_{Po}$$

On the basic of above facts, option (a) is correct.

SECTION B-(Only for Ph.D. candidates)

Q2. A spin-2 nucleus absorbs a spin-1/2 electron and is then observed to decay to a stable nucleus in two stages, recoiling against an emitted invisible particle in the first stage and against an emitted spin-1 photon in the second stage. If the stable nucleus is spinless, then the spin of the invisible particle is

(a) $\frac{3}{2}$ or $\frac{5}{2}$ (b) $\frac{3}{2}$

(c) $\frac{1}{2}$ or $\frac{3}{2}$ (d) $\frac{1}{2}$

Ans.: (a)

Solution.:

$$\begin{array}{ccccccc} X & + & e^- & \longrightarrow & Y & + Z & + \gamma \\ \vec{2} & & \frac{\vec{1}}{2} & & \vec{0} & \vec{S}_f & \vec{1} \quad Spins \end{array}$$

$$\Rightarrow \vec{S}_i = \frac{\vec{3}}{2} \text{ or } \frac{\vec{5}}{2}$$

(a) Let $\vec{S}_f = \frac{\vec{3}}{2}$ or $\frac{\vec{5}}{2}$

$$\vec{S}_i = \frac{\vec{3}}{2} \to \vec{S}_f = \frac{\vec{3}}{2} \to \vec{S}_\gamma = \vec{0}, \vec{1}, \vec{2}, \vec{3}$$

$$\vec{S}_i = \frac{\vec{3}}{2} \to \vec{S}_f = \frac{\vec{5}}{2} \to \vec{S}_\gamma = \vec{1}, \vec{2}, \vec{3}, \vec{4}$$

$$\vec{S}_i = \frac{\vec{3}}{2} \to \vec{S}_f = \frac{\vec{3}}{2} \to \vec{S}_\gamma = \vec{1}, \vec{2}, \vec{3}, \vec{4}$$

$$\vec{S}_i = \frac{\vec{3}}{2} \rightarrow \vec{S}_f = \frac{\vec{5}}{2} \rightarrow \vec{S}_\gamma = \vec{0}, \vec{1}, \vec{2}, \vec{3}, \vec{4}, \vec{5}$$

So option (a) is correct.

(b) It is also correct but option (a) is best possible (c) and (d)

$$\vec{S}_i = \frac{\vec{5}}{2} \rightarrow \vec{S}_f = \frac{\vec{1}}{2} \rightarrow \vec{S}_\gamma = \vec{2}, \vec{3}$$

So, option (c) and (d) are wrong.

Q3. Which of the following decays is forbidden?

(a) $\pi^0 \rightarrow \gamma + \gamma$

(b) $K^0 \rightarrow \pi^+ + \pi^- + \pi^0$

(c) $\mu^- \rightarrow e^- + \nu_e + \bar{\nu}_\mu$

(d) $n^0 \rightarrow p^+ + e^- + \bar{\nu}_e$

Ans.: **(c)**

Solution.:

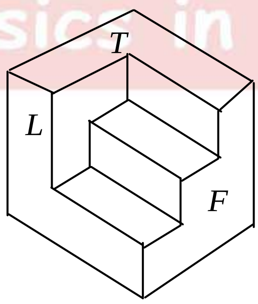
	$\mu^- \rightarrow$	$e^- +$	$\nu_e +$	$\bar{\nu}_\mu$	
Q	-1	-1	0	0	Conserved
L	$+1$	$+1$	$+1$	-1	Conserved
B	0	0	0	0	Conserved
S	0	0	0	0	Conserved
Spin	$\frac{\vec{1}}{2}$	$\frac{\vec{1}}{2}$	$\frac{\vec{1}}{2}$	$\frac{\vec{1}}{2}$	Conserved Conserved
Parity	$+1$	$+1$	$+1$	-1	Conserved
L_e	0	$+1$	$+1$	0	Not Conserved
L_μ	$+1$	0	0	-1	Not Conserved

So this decay is forbidden.

General Physics

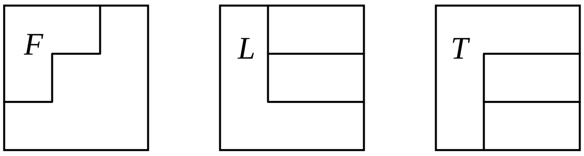
SECTION A-(For both Int. Ph.D. and Ph.D. candidates)

Q1. A three-dimension view of a solid is sketched below:

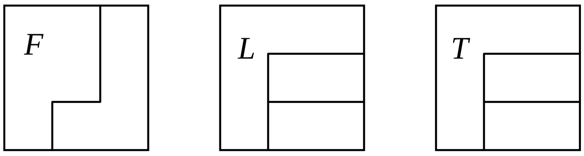


The three projections below are each intended to show the solid from its front (F), left side (L) and top (T), as marked in the figure. Which one is correct?

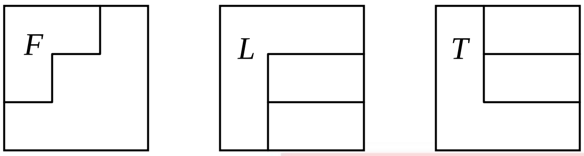
(a)



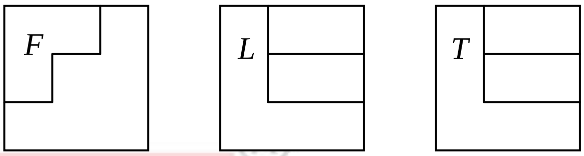
(b)



(c)



(d)



Ans. : (a)

SECTION B- (only for Int.-Ph.D. candidates)

SECTION B-(Only for Ph.D. candidates)

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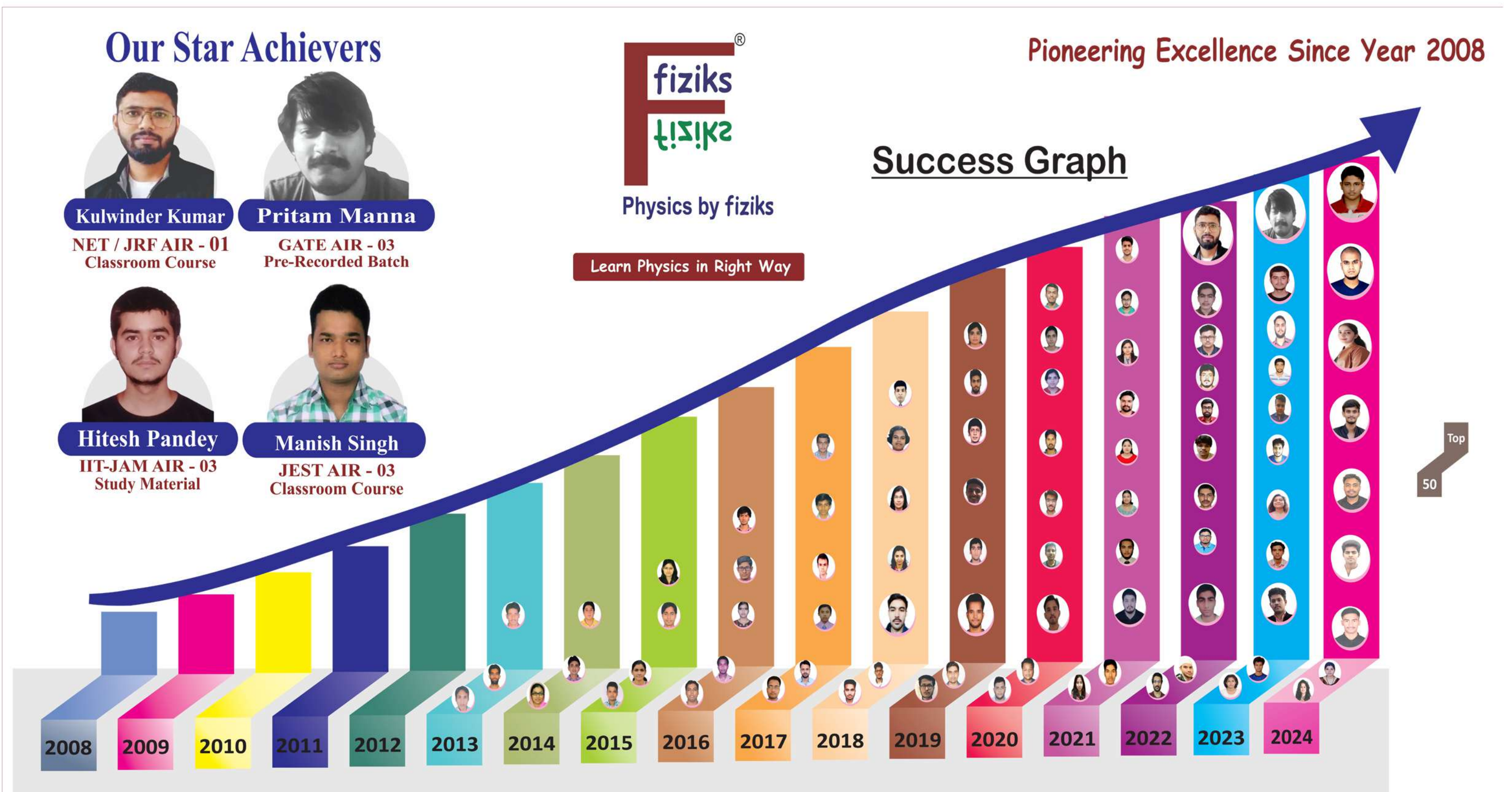

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